Pushing Einstein's Principles to the Extreme *

Gerhard Mack
II. Institut für Theoretische Physik der Universität Hamburg

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In these lectures I propose to push Einsteins principle of coordinate independence to the extreme in order to restrict the possible form of fundamental equations of motion in physics. I start from nearly tautological system theoretic axioms. They provide a minimal amount of *a priori* structure which is thought to be characteristic of human thinking in general. It is shown how formal discretizations of Maxwell and Yang Mills theory in flat space and of general relativity in Ashtekar variables fit into this frame work.

1 What distinguishes truly fundamental physical theories?

The purpose of science is complexity reduction. We wish to understand a multitude of emergent phenomena starting from few basic principles. This tells us what criterion could be used to distinguish between more and less fundamental physical theories. A theory will be the more fundamental the less structure is assumed *a priori*. This is plausible, because what is assumed is not explained. Among the *a priori* structure will be all the axioms of the mathematical theories that are used to formulate the theory. Typically such axioms are relations between mathematical objects, and there is no *a priori* reason why the relations postulated in some arbitrary mathematical theory should find their correspondence in nature.

What would be the minimum *a priori* structure which we need to assume in order to build on it a theory of the world? Certainly the structure of human thinking will need to be included, because we cannot avoid using it in building our theories. Could such an assumption be enough by itself? We do not have the power to prove that it is, but it is very interesting to examine the question of how far one can go, and it brings us into contact with the whole history of human thought.

Philosophers might object that "structure of human thinking" must mean logic, and Immanuel Kant had proven in his famous "critique of pure reason"

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that a theory which describes observed phenomena in the world cannot be deduced from logic alone.

However, studies of linguists [5] (and everyday experience) reveal that the structure of human thinking is not adequately represented by logic.

Moreover, Kant assumed too much structure *a priori*, including his version of the Aristotelian categories, [1] and this leads to too many possibilities of what can be thought of. Einsteins general relativity principle amounts to postulating the *absence of a priori structure* as we will see. It is well known that this relativity principle restricts the form of the possible equations of motion very much. Here I try to push this principle to the extreme.

And last not least, the theory of complex systems has an entirely new selection principle, inherited from quantum field theory and unknown to Kant, by which to select from all the theories which can be thought of those which give rise to *emergent phenomena* that could in principle be observed. I will come back to this later.

One more objection must be answered. It might be argued that assumptions on the structure of human thinking are inappropriate as *a priori* assumptions, they should be deduced from neurophysiological data [16]. But "deduced" is the wrong word. According to iron rules of logic, one is not allowed to use in a deduction that which is to be deduced. Building a theory of mental activity based on neurophysiological data makes use of the structure of human thinking. One is trying to construct a self-consistent picture, and this can certainly produce very important insights, but it is not a deduction, and one *cannot* transcend to a level more fundamental than the structure of human thinking.

I will first state my basic assumption about the structure of human thinking informally. It will be made precise in the next section.

Pre-Axiom: *The human mind thinks about relations between things or agents.* [2]

Relations will be regarded as directed. The traditional notation in logic is aRb for a relation R of a to b. I prefer to use the notation which is now customary in mathematics (category theory), where one denotes objects (things, agents) by capital letters X,Y,... and arrows (relations) by small letters $f, g, ...$, and where $f : X \mapsto Y$ stands for a relation from X to Y.

It will be assumed as a defining property of relations[3] that they can be

[1] In Hofstadters book "Gödel, Escher, Bach" an intriguing interpretation of the notion of enlightenment in Zen-Buddhism is proposed. Briefly it amounts to transcending below the level of Aristotelian categories to a more fundamental level of mental activity - thinking free of (Aristotelian !) categories. In a way we are attempting something like that when we start from general systems and name things which would belong to different categories. The Aristotelian categories are not *a priori* here, in contrast to Kant. In particular, the properties of *space* are not a priori

[2] There exists now a data base which categorizes and lists thousands of things the human mind thinks of, and the relations between them. It was produced by Cycorp corporation with a view to commercial applications, see the entries "The Cyc Technology" and "The Upper Cyc Ontology" under http://www.cyc.com , especially http://www.cyc.com/cyc-2-1/toc.html (status Sept 96).

[3] It is said that string theory tries to construct geometry from extended objects. But what

composed. If there is a relation $f : X \mapsto Y$ from X to Y and a relation $g : Y \mapsto Z$ from Y to Z, then this defines a relation, denoted $g \circ f : X \mapsto Z$ from X to Z. Think of a friend of a friend, or of a brother in law which is the husband of a sister. There can be relations from X to X; among them is the identity ι_X of X with itself.

Typically, a relation $f : X \mapsto Y$ from X to Y specifies relation in the opposite direction, denoted $f^* : Y \mapsto X$. If X is the wife of Y then Y is the husband of X.

The objects of a system can themselves be systems, i.e have internal structure. In this way, a general frame work for the discussion of self organization in complex systems is obtained.

A generalized notion of locality will be built into the axioms. We know since the discovery of Faraday's Nahewirkungsprinzip in the last century that fundamental physical laws relate only physical quantities at infinitesimally close points of space time. In discretized theories, the notion of infinitesimally close is replaced by nearest neighbor relations; this specifies a graph (e.g. a lattice) and singles out certain relations as fundamental. All other relations can be composed from fundamental ones. The fundamental relations will be called *links*.

This notion of locality leads to a definition of the notion of *emergence* which is a key concept of complex systems theory. Emergence is the appearance of nonlocal phenomena as a consequence of local laws. Propagation of electromagnetic waves is an example, and also the reproduction fork dynamics which models the replication of DNA in cells (see ref. [8] and figure 9 below).

These basic assumptions will be subsumed in the axiomatic definition of a *system*; mathematically it is both a category and a graph. The relations are the arrows of the category, and there is a *-operation on arrows.

The objects are actually of secondary importance. They can be reconstructed when one knows which arrows can be composed, and what is the result of the composition.

Classically, the state of the world at one time (and also the world *sub specie aeternitatis*) is assumed to be described by a system of this kind. Quantum mechanically, there is a wave function which assigns a complex amplitude to systems.

There are no numbers in this to begin with, and no arithmetic operations. One cannot make mistakes of 2π in the fundamental equations because there is no π. The only substitute for arithmetic operations is the composition $\circ$ of relations. As a result, truly fundamental physical laws - those that can be stated in this language - cannot contain any (dimension-less) free constants. Also one cannot "add" physical theories (e.g. Einstein + Maxwell) in a familiar way.

It will be seen later how one comes to correspondences with quantitative theories in the first place.

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means "extended" before there is space? It can only mean a property which is going to be interpreted as "being extended" after space has been constructed. In the present frame work, *being a relation* is such a property.

Coordinates are numerical encodings of positions in some space. The absence of *a priori* numerical structure is a way to push coordinate independence to the extreme.

How does one build a theory of the world on so little *a priori* structure? It proceeds in two steps

1. Name things,
2. Make statements about named things.

In this paper I will be chiefly concerned with fundamental physics. The "things" which will be named and examined will be

electro-magnetic fields and Yang Mills fields,

space (in the sense of space-like hyper-surface of space time),

matter (Dirac fields).

To explain the naming step, it is necessary to distinguish between two different types of physical laws in the traditional formulation.

First, there are laws which constrain the state of a part of the world at one time. Gauss' law in electrodynamics is a most important example of such a law. In a canonical formalism these laws are called *constraints*. All the fundamental physical theories, including general relativity, are gauge theories, and they all obey nontrivial constraints. There are further properties which can be read off the state at one time, and which are preserved in time. I will count them among the constraints. It will be seen that our *a priori* structural assumptions, as poor as they are, provide for a gauge group (or a substitute for it) which can be read off the initial state, and for a notion of gauge invariants which determines what could be observable in a particular kind of system. The named things in the above list will be systems which are distinguished by the validity of constraints which are characteristic for them. The statement of the constraints must be meaningful, given only the *a priori* structure which is furnished by the axiomatic properties of a system.

One may ask the philosophical question whether the constraints are really physical laws, or just denominations. This brings us back to the discussion above. The principle of emergence may single out some of the possible constraints as physical laws because such properties of systems are the only ones which can be observed at a macroscopic level.

Secondly there are laws which govern the dynamics (time development) of a system. I will seek dynamical laws which are universal in the sense that they can be stated in a meaningful way for any system whatever. This is a very restrictive requirement on a truly fundamental dynamical law, because there is so little *a priori* structure which can be used to write down an equation of motion.

In this paper I concentrate on the conceptual issues. The precise form of the equations is still open to experimentation. There is an essentially unique first order equation of motion, cp. figure 2. To accommodate second order dynamics, I admit two different kinds of links - essentially coordinates and momenta (or velocities). They are represented by thin and fat lines in the figures. But this weakens uniqueness; unfortunately there are now several equations which can be written down. Universal forms of the equations of motion of Maxwell,Yang

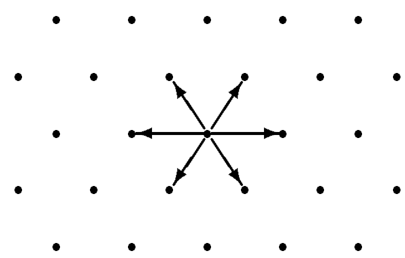

Figure 1: triangular lattice

Mills and of Einstein are shown in figures 3 and 5. One sees that they fit on the template, figure 2, but they are not exactly the same. The constraints, figures 4, 6 are also not the same, but this is as it should be.

It remains to be seen whether the two kinds of links can be fused into a single one [4] which satisfies one single universal equation of motion. The universal law is supposed to specialize to the known fundamental dynamical laws when applied to states which satisfy the appropriate constraints.

There is no rigorous classification yet, and the investigation up to now are at the level of formal discretizations of known continuum theories. The indications from the available evidence are that the following theories admit a universal formulation in the system theoretic frame work

1. General relativity with or without massless Dirac matter fields

2. Yang Mills theory in flat space with or without massless Dirac matter fields

But Einstein Maxwell theory (or Yang Mills theory in curved space) does not appear to fit; it is not unified enough. Also a cosmological constant, fundamental masses or fundamental Higgs fields do not fit. The problem comes from explicit factors $\sqrt{(-g)}$ and $g^{\mu\nu}$ which cannot be absorbed. I will describe below a universal formulation of Maxwell- and Yang Mills equations, of the Einstein equations, and of the massless Dirac equation.

Discretizations of super-symmetric theories have not been investigated yet. They ought to be investigated because they may offer the best chance of leading to emergent phenomena by virtue of cancelations of divergences at short distance which one encounters when one tries to enforce long range effects of short range interactions.

1.1 Einstein's principles

Let me pause to discuss how Einstein's principles fit in with the philosophy.

The two underlying principles of Einstein's General Relativity are the principle of relativity, or general covariance, and the equivalence principle. When appropriately interpreted these principles are also operative in the gauge theories of elementary particle physics (modulo troubles with the Higgs sector)[4]. It is well known how these principles constrain equations of motion.

[4] In the Poincaré gauge theory approach to general relativity one tries to achieve such a fusion by interpreting vierbeins as vector potentials of the translation group. [26]

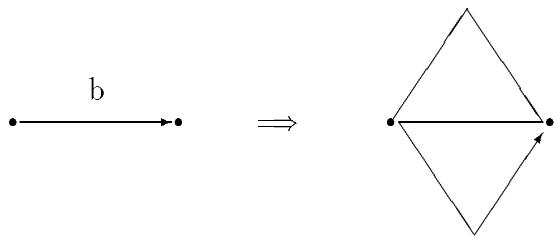

Figure 2: The universal equation of motion of fundamental physics. The symbol $\Longrightarrow$ symbolizes the effect of one time step. There is a product over all triangles which share the link b^* ($= b$ with opposite orientation). A variant of the equation exists which has the orientation of the triangles reversed. The gauge covariant massless Dirac equation is a special case; it governs the evolution of links b to or from ∞.

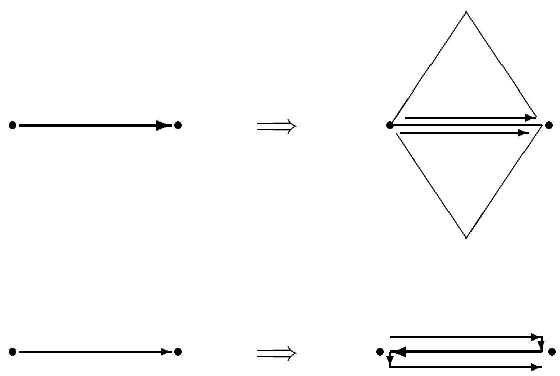

Figure 3: Maxwell Equations of Electrodynamics. The Yang Mills equations of general gauge field theories have the same form. It involves a product over all triangles which share the horizontal link. In the presence of Dirac matter, the triangle can have a tip at ∞.

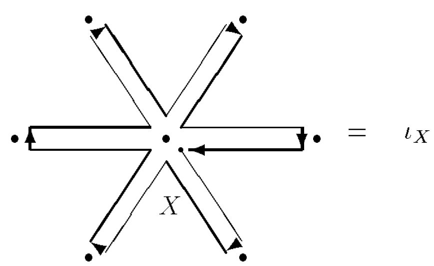

Figure 4: Gauss law for Electrodynamics, Yang Mills theory and General Relativity. In the presence of Dirac matter, one of the points is at ∞.

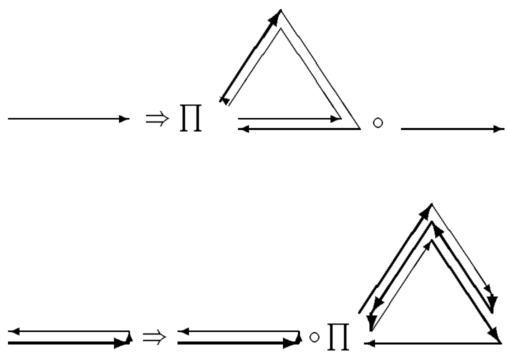

Figure 5: Equations of motion of general relativity. The product is over all triangles which share the horizontal link.

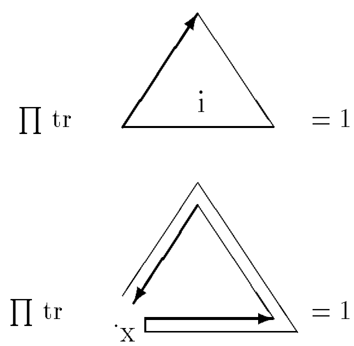

Figure 6: vector and scalar constraint of general relativity. There is a product over triangles which share the link i and the corner x, respectively. In $2+1$ dimensions there is a simpler version, cp. figure 7.

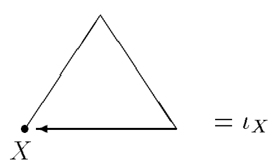

Figure 7: Simplified form of vector + scalar constraint for $2+1$-dimensional gravity. The equality must hold true for every thin-lined triangle

The principle of relativity is a statement of *absence of a priori structure.* Before general relativity it was thought that space has an *a priori* structure which defines the notion of a straight line. This is equivalent to an *a priori* defined possibility of comparing directions at different points in space. This *a priori* structure is abandoned in general relativity and in gauge theory. To compare tangent vectors (or vectors in color space) at different points of space time one must use parallel transport of vectors from one point to the other, and the result depends both on the path along which one transports, and on a connection ($Sl(2,\mathbf{C})$-gauge field in relativity) which is *dynamically* determined as a solution of equations of motion. Gauge covariance follows.

In the traditional formulation of general relativity, the principle of relativity is not pushed to its logical conclusion, though. The assumption of an *a priori structure* of space time as a differentiable manifold means that one assumes an a priori definition of straight line *in the infinitesimally small.* It has been suspected for a long time that this is an unreasonable assumption when it comes to physics at the Planck scale. In the traditional formulation, general covariance demands that there should be no preferred coordinate system. But one assumes an *a priori* defined preferred *class* of coordinate systems, viz. smooth coordinates. General covariance is then interpreted to mean that the fundamental equations of the theory retain their form under transformations of coordinate systems *within the preferred class.* If one pushes the principle of coordinate independence to its logical conclusion, *the fundamental equations should make sense without any reference to coordinates whatever.*

The principle of equivalence asserts that the motion of material bodies is free in a local Lorentz frame. The notion of free motion makes essential reference to an *a priori* defined notion of straight line in the infinitesimally small. But when matter is described quantum mechanically, the notion of "straight ahead" in the infinitesimally small is no longer needed. Newton's law gets replaced by a Schrödinger equation which involves a gauge invariant Laplace or Dirac operator. To define it one needs only the appropriate parallel transporters, plus linearity which is supplied by the principles of quantum mechanics. In conclusion it is reasonable to hope that a sufficiently strong principle of coordinate independence alone should be sufficient to single out the truly fundamental dynamical laws in physics.

Along the way, principal fiber bundles will go away. Mathematical physicists tend to think that principal fiber bundles are the essence of gauge theory, but it is not so. The definition of a global multiplication from the right with elements of the structure group is is an a priori *global* structure. It amounts to postulating certain invariants. It is contrary to the spirit of gauge theory which emphasizes locality. The principle fiber bundle structure provides for an *a priori* tensor product of representations which commutes with parallel transport. But different tensor products are used in gauge theories with quantum gauge groups [6]; they are used in some models of quantum space time [7].

2 System theoretic foundations

Motivated by the pre-axiom of the previous section I will now give a formal definition of a system. Following the terminology in category theory, the agents of a system will be called *objects* and the directed relations between them are called *arrows*.

A *complete system* consists of *objects* X and *arrows* $f : X \mapsto Y$. X is called the *source* and Y the *target* of the arrow f. Some of the arrows are declared fundamental and are called *links*. They are typically denoted by b. The following conditions are imposed.

1. Arrows can be composed. With $f : X \mapsto Y$ and $g : Y \mapsto Z$, the arrow $g \circ f : X \mapsto Z$ is defined. Composition $\circ$ is associative.

2. To every object X there exists a fundamental arrow $\iota_X : X \mapsto X$, called the identity arrow; $\iota_Y \circ f = f = f \circ \iota_X$ for every arrow $f : X \mapsto Y$.

3. All arrows can be obtained by composing fundamental ones
$$f = b_n \circ ... \circ b_2 \circ b_1. \tag{1}$$

4. To every arrow $f : X \mapsto Y$ there is an adjoint arrow $f^* : Y \mapsto X$ such that $f^{**} = f$ and $(g \circ f)^* = f^* \circ g^*$, $\iota_X^* = \iota_X$.

5. The graph whose vertices are the objects and whose links are the fundamental arrows is connected. Equivalently: For every pair X, Y there is an arrow $X \mapsto Y$

Mathematically, a complete system is both a category and a directed graph. The first two axioms are those of a category, the fourth one asserts the existence of adjoints of arrows. To specify the graph, certain arrows are singled out as links. The objects are the vertices of the graph and the fundamental arrows are the links of the graph. To every link, there is a link in the opposite direction, but this requirement will be abandoned.

In a *system*, the 4-th axiom is relaxed. I admit that this is motivated by hindsight. It is necessary to accommodate certain dynamical processes which are very important in biology, such as DNA-replication [8]. The adjoints of some of the fundamental arrows b are allowed to be absent. Thus, a *system* can be thought as being obtained from a *complete system* by declaring certain links as absent, and with them all arrows which can no longer be composed from fundamental arrows. However, the absent adjoints can be added again in a unique fashion.

We write $b^* = 0$ if b has no adjoint. There are now two possibilities
$$b^{**} = b \quad \text{or} \quad b^{**} = 0. \tag{2}$$

Given a category, one needs to single out links to get a graph and a system. Let us examine the converse question: Given a directed graph, to what extent does it specify the system?

We can define a path from $X = X_0$ to $Y = X_n$ to consist of a sequence of links $b_0, b_1, ..., b_n$ such that the target of $b_i : X_i \mapsto X_{i+1}$ is also the source of b_{i+1}. Paths can be composed by juxtaposition. Adding the identity links, we obtain in this way a category and thereby a system $\bar{S}(G)$ in a canonical way from a graph G. Therefore the assumption that relations can be composed is in fact a tautological one. It is useful because it institutionalizes the possibility that different paths may represent indistinguishable relations, as follows.

Suppose we start from a system S, and we reconstruct from its graph G the system $\bar{S}(G)$. It needs not be equal to S because different paths may define the same arrow in S. The arrows in S are in general equivalence classes of paths. Therefore, given a graph G, a system can be defined by specifying a *generating set of relations between links.* In the spirit of our locality principle, local relations are particularly important.

Two most important examples of such relations are as follows

$$b^* \circ b = \iota_X \quad , \quad b \circ b^* = \iota_Y , \tag{3}$$

$$b_3 \circ b_2 \circ b_1 = \iota_X \tag{4}$$

for all links $b : X \mapsto Y$ and for all triangles (i.e. loops of three links) from X to X, respectively. Interesting generalizations will be encountered when we come to the Dirac equation. They differ only in some -signs.

A further interesting type of relation is

$$\epsilon \circ b^* = b^{-1} \circ \epsilon \tag{5}$$

where $\epsilon : X \mapsto X$ are *square roots of* $-$ *signs.* This could be used to characterize $sl(2, \mathbf{C})$ connections as appear in general relativity.

In the system theoretic frame work, a $-$*sign* is a collection of links, denoted $-\iota_X : X \mapsto X$, such that $-\iota_X \neq \iota_X$, but $(-\iota_X) \circ (-\iota_X) = \iota_X$, and $(-\iota_Y) \circ b = b \circ (-\iota_X)$ for all links $b : X \mapsto Y$.

A system will be said to be *unfrustrated* if there is at most one arrow from X to Y, whatever X, Y. Curvature in general relativity and field strength in gauge field theory are instances of frustration.

We will also need a notion of isomorphism of systems because we will not distinguish between isomorphic systems.

A *functor* $\mathcal{F}$ is defined as in category theory. It is a map from one system to another one which preserves identity and composition law. If $f : X \mapsto Y$ then $\mathcal{F}(f) : \mathcal{F}(X) \mapsto \mathcal{F}(Y)$, $\mathcal{F}(\iota_X) = \iota_{\mathcal{F}(X)}$, and $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$. It is not required that $\mathcal{F}$ maps fundamental arrows into fundamental arrows, but it is postulated that $\mathcal{F}(f^*) = \mathcal{F}(f)^*$.

Such a functor is called an *isomorphism* of the system if it has an inverse functor, and if it maps fundamental arrows into fundamental arrows.

2.1 The language of thought

Our assumptions on the structure of human thinking amounts to the postulate that the human mind manipulates objects and relations of systems by operations which are well defined as a consequence of the axiomatic properties of a system. If one entertains the notion that thinking uses some sort of language, then one would be lead to calling the system theoretic frame work the *language of thought*. However, it is different from natural languages and from artificial languages including formal systems in one crucial aspect. All true languages have a serial structure. They are modeled on verbal utterances which are one word after another. General systems have no serial structure.

Two questions arise naturally. How do properties of systems which occur in our mind during mental activity get translated into statements of a natural language, and where in the brain does the translation take place?

Related questions were raised by Raichle [14] in his interpretation of recent neurophysiological experiments which localize types of mental activity in the brain by a differential measurement of blood flow using PET (or Nuclear Magnetic Resonance).

3 Gauge theory aspects

I tried to make precise the idea that the human mind thinks about systems which consist of things and relations between them. It will presently be seen that this encapsulates the essence of gauge theories as we know them in physics, in spite of the poverty of the assumed *a priori* structure.

Let G_X consist of all arrows $g : X \mapsto X$. They are called loops. Because of the composition law, G_X is a semi-group. It will be called the holonomy semi-group or local gauge semi-group at X.

A gauge transformation is a map of the system which takes every object X into itself, and arrows $f : X \mapsto Y$ into new arrows $f' : X \mapsto Y$ such that

$$g(Y)f = f'g(X) \tag{6}$$

for all arrows f and a suitable choice of $g(Z) \in G_Z$ for all Z. Such a map is automatically functorial, [5] i.e.

$$(g \circ f)' = g' \circ f'$$

In unfrustrated systems, the gauge semi-groups are trivial, i.e. they consist only of the identity ι_X.

In our physical application, each arrow $b : X \mapsto Y$ will have an inverse b^{-1} such that $b \circ b^{-1} = \iota_Y$ and $b^{-1} \circ b = \iota_X$. In this case the local gauge semi-group is actually a group, and it is independent of X modulo isomorphism. It is called the *gauge group*. Gauge transformations take the familiar form

$$f' = g(Y)fg(X)^{-1}$$

Let us consider some examples of systems.

[5] In category theoretical language it is a functor which preserves objects and which admits a natural transformation to the identity[11].

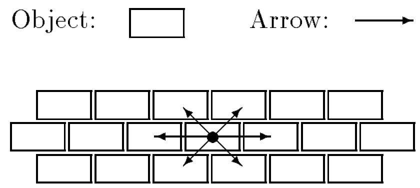

Figure 8: The structure of a brick wall

Example 1 (triangulated manifold) *Consider a triangulated manifold. The objects of the system shall be the 0-simplices, and the links the 1-simplices. The adjoint link is given by the 1-simplex with opposite orientation. Imposing the two relations (3),(4), the arrows $f : X \mapsto Y$ of the system will be the homotopy classes of paths from X to Y. The gauge group is the fundamental group of the manifold. The system is unfrustrated if and only if the manifold is simply connected*

Example 2 (brick wall, see figure 8). *The objects are the bricks, and the fundamental arrows are the translations which take one brick to the position of a nearest neighbor. They can be composed to translations to other bricks positions. The system is unfrustrated, and the gauge group is trivial.*

Example 3 (logical archetype) *The system has two objects, denoted T and F and three fundamental arrows other than the identities:*

$$e : T \mapsto F, \quad e^* : F \mapsto T, \quad o = o^* : F \mapsto F$$

subject to the relations

$$e \circ e^* = \iota_F, \; e^* \circ e = \iota_T, \quad o \circ o = o.$$

The gauge semi-groups for T and F are isomorphic to the two element semi-group $\{\iota_F, o\} \cong \{1, 0\}$ with the usual multiplication law.

3.1 Representations

I will introduce a general notion of representation.

In group theory, a representation is not simply a homomorphism from one group to another. It is required that the representation operators are linear operators in a Hilbert space. As a result, there is an a priori defined multiplication for them which is consistent with the linear structure in the Hilbert space in the sense that the distributive law holds.

Similarly, models in model theory [10] are also a kind of representation. They are structure preserving maps whose images are sets.

And in everyday life, an oil painting has some a priori structure in addition to representing the structure of what is painted - it consists of paint of some chemical composition on canvas.

Motivated by this, *representations* of a system S will be defined as functorial maps into some given system or into an element of a class of systems which come equipped with some characteristic additional structure. It is required that fundamental arrows are taken into fundamental arrows.

Example (logical representations) *Equipped with a binary product* $|$,

$$T|T = F, \quad T|F = F \quad F|F = T \tag{7}$$

the logical archetype (example 3 above) appears as the image of representations $\mathcal{F}$ *of systems which come equipped with a product (binary composition)* $|$ *of objects. The notion of a product is understood to demand that there are links* $A \leftarrow A|B \rightarrow B$. [6]. *Objects are interpreted as propositions, links* $A \mapsto B$ *(other than the identity) are interpreted to mean " A excludes B", and* $|$ *means "neither nor". If a link's adjoint is its inverse, it gets interpreted as negation. A representation assigns a truth value* T *or* F *to every object (proposition); the representation property ensures that the rules of logic are obeyed, provided it is required that the representation preserves composition,* $\mathcal{F}(A|B) = \mathcal{F}(A)|\mathcal{F}(B)$, *and the special links* $b : A|A \mapsto A$ *are unitary (i.e. they obey* $b^* = b^{-1}$*)) so that they will be interpreted as negation.*

Such representations may exist or not, and they may be unique or not. One may consider logical representations of arbitrary systems with a product (binary composition $|$*) but it is natural to require unitarity* $b^* = b^{-1}$ *of the special links* $b : A|A \mapsto A$. *One writes* $\neg A = A|A$. *If there is a subsystem of the form* $\neg(A|\neg A) \mapsto B$ *then* $\neg B$ *is interpreted as an axiom, because in any representation* $\neg(A|\neg A)$ *is true.*

A representation may fail to exist because the axioms are contradictory. If the representation is not unique, then the truth of some propositions cannot be decided from the axioms.

3.2 Representation of a system as a communication network

Next I will state a representation theorem which will show that in spite of the nearly tautological character of our assumptions, all the essential structure of lattice gauge theory (on irregular lattices) is encapsulated in it, *except* for the linearity of the charge - or color spaces whose elements are subject to parallel transport. The arrows will become maps, but not necessarily linear maps.

Representation theorem: *Every (finite) system admits a faithful representation as a network as follows: There are spaces* Ω_X *associated with objects* X *and arrows act as maps* $f : \Omega_X \mapsto \Omega_Y$, *with* $\iota_X = id$.

The construction of the space Ω_X uses the sets of all arrows to and from X. Details are given in Appendix A.

For now let us talk of one time. Then the maps f may be interpreted as channels of communication. Time development (and acts of communication) is only considered later.

[6] This generalizes a corresponding construction in category theory [11]

The sets Ω_X need not be linear spaces and the maps f need not be linear. *Apart from this, the setup is as in lattice gauge theory..* The objects X may be elements of an irregular lattice but irregular lattices were considered before.

Scholium:(Lattice gauge theory) [12] *In the Hamiltonian formulation of lattice gauge theory, space is a discrete lattice like figure 1. In the continuum, one has vector potentials $A = A_i dx^i$ which take their values in the Lie algebra of the gauge group. From them, parallel transporters along paths C in space from X to Y are constructed as path ordered products $u(C) = P\exp(-\int_C A)$. They map the fiber Ω_X of a vector bundle at X into the fiber Ω_Y at Y. In lattice gauge theory, the parallel transporters along the links of the lattice are the basic variables of the theory. Finite difference versions of covariant derivatives are constructed with the help of these parallel transporters.*

Values of matter fields $\Psi(X)$ could be interpreted as elements of Ω_X but we will prefer to regard them as maps (i.e. links) from some "flavor space" Ω_∞ to Ω_X in the later discussion of the Dirac equation.

The equations of motion and Gauss' law are the same as in the continuum, except that finite difference covariant derivatives in space are to be used. One may go on to discretize also time, which means that also time derivatives get discretized.

Let me emphasize that the representation theorem constructs a space Ω_X for every object X *but it does not attribute a state $\xi \in \Omega_X$ to the objects.* The objects have no state. Dynamics consists of structural transformations, not of changes of states of objects. This is a big difference to cellular automata. Nevertheless there is a connection. At the level of *effective theories* which operate on larger scales, the objects can be systems themselves, and so they have internal structure. Changes of this internal structure could be interpreted as changes of a state of the object.

It is appropriate to cite also the computer pioneer Konrad Zuse's work on "computing space" [28] for similarity in spirit. The frame work which I use here to discuss fundamental physics is also employed as a tool in massively parallel computing [8].

Let me clarify that I do not regard values of matter fields $\Psi(X)$ in gauge field theories as elements of Ω_X because otherwise it would be impossible to find a universal equation of motion for the links in which the values of the matter fields would enter. This brings us to the next topic.

4 Universal dynamics

Next we turn to the time development $t \mapsto S_t$ of a system. Sorin Solomon proposed to call it "drama". It is supposed to be governed by an equation of motion.

According to our guiding principle, the most fundamental equations of motion should have the property that they can be formulated purely within the frame work provided by the language of thought, without need for any further a priori structure. In other words, they should be meaningful for every system whatever. I will call this a universal dynamics.

Another consideration leads also to universal dynamics: A state should contain all necessary information about its time development in itself, without need for further extrinsic specification. Different kinds of systems should be distinguished by different properties of the initial states.

Gauge invariance is automatic in a universal dynamics because there is no intrinsic way to distinguish between isomorphic representations of a system.

I will consider dynamics in discrete time. Dynamics in continuous time would require some assumptions of *a priori* structure such as spaces Ω_X which are manifolds.

I will assume at first that the dynamics is of first order, so that the system S_t at time t determines the system S_{t+1} at time $t+1$. Generalization to second order dynamics will be considered later and reduced to the first order case, but with two kinds of links.

In the spirit of the discussion of locality in section 2 it is demanded that the dynamics is local in the following sense

Every object is descendent of some object X and every link is descendent of some arrow f of the system one time step ago. Descendents of X are determined by X and by the fundamental arrows of X alone. Descendent links of f are determined by f if f is fundamental, by source X and target Y of f, and possibly by the fundamental arrows to and from X or Y.

The formulation of a dynamics is a rule how a new system is to be made out of a given one. It is supposed to have the stated locality properties. The possibilities of formulating such rules within the language of thought are very restricted. In fact, the innocent looking assumption of a system with a *finite* number of links has introduced a priori structure of countability. I exorcize it again by not admitting the possibility of counting the number of links to an object. It should make no difference if several simultaneous links from X to Y are regarded as one link.

Basically there are three kinds of change with time, apart from death.

1. Growth
2. Motion
3. change of composition law

This classification applies not only to material bodies in space, but in this paper we are only concerned with physics.

I speak of growth if there is copying of objects or of links, or if adjoints b^* of links b are newly produced. There can also be fusion of isomorphic subsystems under some conditions. The aforementioned reproduction fork dynamics is a universal dynamics. It models DNA replication [2] but has also much more general copying capabilities. It is a local dynamics which propagates a copy-process in such a way that systems of completely arbitrary topology can be copied. It is shown in figure 9. For further explanation see ref. [8]. Locality is important because enzymes in a biological system act locally.

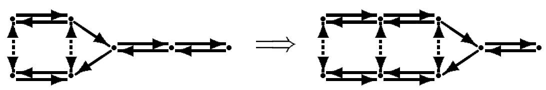

Figure 9: Reproduction fork dynamics - a universal copy machine for systems. A pair of links without adjoints to and from an object X is called a fork. The presence of a fork causes X to be copied. The bidirectional links get split to become forks and the two halves are divided among the copies of X. The links which had no adjoint before get one. Once a copy process is started at some initial object X_0, the forks travel through the whole system and one gets two copies of the system as a result. (The dotted arrows are only there to indicate the fact that the objects are copies of each other.) This works for systems of completely arbitrary topology.

In principle there exists the possibility of a change in the composition law $f \circ g \Rightarrow f \circ s \circ g$, where s is a loop. But I will not enter into a discussion of this possibility here.

Here I will be chiefly interested in motion. It consists of changes of arrows. [7] This includes changes of relations of an object X to itself; these relations could be regarded as properties of X.

The possibilities are very limited. How can a link $b' : X \mapsto Y$ of the system at time $t+1$ arise? It can only have been composed from links of the system S_t at time t. (Creating new links e.g. by taking adjoints would be regarded as growth). But all that can be made by composing links is an arrow of the category. So the rule has the form

$$b' = f \tag{8}$$

where f is a possibly composite arrow of S_t. But this means that *the category does not change at all.* [8] The only change is in the specification which arrows are considered as fundamental. *Motion means that composite relations are declared fundamental.* One can think of it as composition of links or bonds by objects which act as catalysts in a manner which is familiar from chemistry, see figure 10.

Such catalysis of relations also plays a basic role in Spinoza's famous treatise on ethics, *Ethica, ordine geometrico demonstrata* [17].

The interpretation of the motion of a point particle in space is shown schematically in figure 11.

Let us return to the analysis of the possibilities for f in the general formula (8) for motion.

[7] As I mentioned before, the objects are secondary. This is so because they can be recovered from the arrows and the composition law according to the representation theorem. This is true up to isomorphism. Isomorphic systems are not regarded as different. There is no intrinsic way of distinguishing between them.

[8] This reminds of Parmenides, the Greek father of ontology. He held that nothing can appear or disappear in the world because this would contradict the nonexistence of the nonexistent. Such changes are only apparent ones to man.

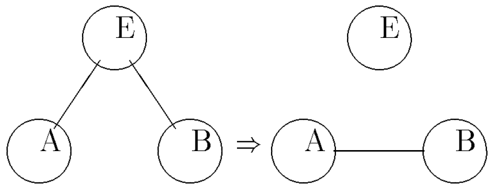

Figure 10: Catalysis in chemistry. Enzyme E binds molecules A and B. First a substrate-enzyme complex is formed where A and B are bound to E. Then the composite link between A and B is transformed into a fundamental one

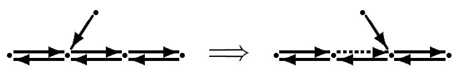

Figure 11: Interpretation of motion as transformation of a composite relation into a fundamental one. The objects which are connected by bidirectional links are interpreted as space points, and the other object as a particle (or as "the idea of matter"). The link from the particle to a space point x represents the relation of "being at x". Motion takes place when a composite arrow made from the relation b of the particle to its former position, and a relation of this space point to a neighbor is declared fundamental, while b loses this status. b remains in the category as a composite arrow.

The links in f are restricted by the locality demand. The simplest possibility is as follows.

$$b \mapsto b \circ b^* \circ b. \tag{9}$$

This reminds of Hegel's dialectic process. Let us follow Hegel in speaking of a "Denkbestimmung" in place of a link. Then the process gets verbally described as follows. A Denkbestimmung (thesis) combines with its opposite (antithesis) to form a new,"more advanced" Denkbestimmung (synthesis). Actually something new is obtained only if b^* is not the inverse of b.

The dialectic process is truly a universal dynamics in the sense of the above definition. But it is not the only one. Neither Maxwell's equations nor Einstein's are of this form.

Other possibilities are found by making use of links b_i to or from X or Y other than the original link $b : X \mapsto Y$ and its adjoint b^*. Assuming, contrary to Hegel, that $b_i^* \circ b_i$ constructions yield nothing nontrivial, the useful links can only occur in triangles $\triangle_i$ which contain b or b^*. The triangular paths have the form

$$\triangle_i = b^* \circ b_i^1 \circ b_i^2. \tag{10}$$

or adjoint of that. This involves links $b_i^1 : X \mapsto Z$, $b_i^2 : Z \mapsto Y$, $(Z \neq X, Y)$. Different triangles cannot be distinguished in an intrinsic way. Therefore they will have to appear symmetrically in the rule. Also we have no way of adding contributions. The only possible composition is with $\circ$. This leaves us with the possibilities

$$b \mapsto b \circ \triangle_1 \circ ... \circ \triangle_n \tag{11}$$

or

$$b \mapsto \triangle_1^* \circ ... \circ \triangle_n^* \circ b \tag{12}$$

where $\triangle_1 ... \triangle_n$ are all the different triangular paths of the form (10).

This is the universal equation of motion of fundamental physics, modulo some complications which we will discuss below.

The path on the right hand side of eq. (11) starts with a factor $b \circ b^*$ which could be omitted.

A schematic graphical representation of eq.(11) is in figure 2.

4.1 Universal conservation law

In the absence of growth processes, a universal equation of motion implies substitutions $b \mapsto f$ which replace links by arrows which existed before in the category. Therefore the category $Cat(S)$ does not change in the course of time. Quantities Q which are determined by the isomorphism class of the category are conserved. In particular the gauge group is time independent.

If there are growth processes, new objects which are copies of old ones may appear, and also new links which are copies or adjoints of old ones. In this case the conservation laws are more subtle.

5 Maxwell's equations

In general, a system is identified as *space* by the validity of certain constraints. This will be discussed below when we come to general relativity. In this section, I am not interested in this aspect and I will assume that we know already what is a discretized flat space. Let us think of a triangular lattice like figure 1.

Maxwell's equations come in two groups. The first group states restrictions on the initial state. The equation $div\, B = 0$ is automatically satisfied through the introduction of a vector potential, and there remains only Gauss law, $div\, E = \rho$. Gauss law in our language is shown in figure 4

The second group contains equations of motion. Their universal version is shown in figure (3).

The Yang Mills equations have exactly the same form, apart from an ordering problem which will be discussed. But the gauge group - which is a property of the initial state - is different. The equations retain their form in the presence of Dirac matter fields, but in this case one of the points in the diagram represents a flavor space (or point at infinity) rather than a space point.

I emphasize that these equations are generally meaningful, but they reduce to Maxwell's or Yang Mills equations only on a regular "flat" lattice. I do not know a universal version of the Maxwell or Yang Mills equations on curved space.

The gauge group is part of the data which specify which particular aspect of the world we are dealing with. In Electromagnetism, the gauge group is isomorphic to $U(1)$. This group admits a natural parameterization by real numbers $0...2\pi$. The magnetic field is a an element of the gauge group (parallel transporter around a loop) and so it acquires a numerical status. Similarly the electric field is a loop which involves parallel transport forth and back at two successive times. In this way they both become "quantities" in spite of the fact that the general frame work knows no numbers. They are gauge invariant and therefore observable in principle.

If the gauge group is noncommutative, there arises an ordering problem. In what order shall the triangles be traversed? The only reasonable answer is "at random". This introduces some stochasticity which may be thought to be a remainder from the quantum theory. In the formal continuum limit its effect disappears.

In the quantum theory, the superposition principle furnishes a commutative operator $+$, and we can sum over triangles instead of composing triangles with the help of $\circ$. So the above ordering problem disappears, but in its place we have the "ordering problems" of quantum mechanics. A universal Schrödinger equation is described in [1] and in section 10 below.

6 General relativity

Next I turn to the equations of motion and constraints of general relativity. The constraints are the properties which a system should have in order to be interpreted as space (in the sense of space-like surface in space time). Among the constraints is the selection of the gauge group $Sl(2,\mathbf{C})$. In 2+1 dimensions it would be the covering group $Sl(2,\mathbf{R})$ of the 3-dimensional Lorentz group instead.

Also among the constraints is the existence of an invariant trace tr which maps loops to numbers.

The remaining constraints are shown in figure 6, and the equations of motion in figure 5. They come from the canonical formulation of general relativity in Ashtekar variables. For the readers convenience a brief review is given in Appendix B.

There exists also a version of the constraints which does not involve tr, cp. [18]. It is only equivalent under invertibility conditions on the dreibein. A similar reformulation exists in 2+1 dimension where it takes the very simple form of a flatness requirement $F_{ij}(x) = 0$. [20] In our language this is eq.(4) (figure 7) for thin links b_i. It says that thin loops around triangles equal the identity.

7 Interpretation of the equations of motion

7.1 Syntax and Semantics: Classification of initial states

The graphical form of the Maxwell, Yang Mills, and Einstein-Ashtekar equations could have been written down by pre-Sokratic philosophers. They must surely have conceived of the idea that the world evolves by rearrangement of triangles. But they could not possibly have found the proper interpretation.

And some readers will no doubt be left in a state of perplexity by the claim that the universal equations of motion (3) contain the Maxwell, Yang Mills equation. The precise correspondence will be explained. But the perplexity itself is worth a comment. It makes it clear that the problem of describing nature and its laws is not solved yet by stating laws in the form of equations. These equations live on a purely spiritual level. They are syntactical rules. In addition one will ask for their meaning. This is the question of semantics. It asks how entities in the equations correspond with phenomena which can be observed in nature. Physicists say the theory itself must determine what quantities are observable. The question about meaning is therefore divided into two questions. The first questions is what quantities from the equations can be observed in principle. The second question is where to find these quantities in nature.

The first question has already been answered. Observations give answers yes or no to questions whether certain statements about a system are true. The permissible statements are those which can be formulated in the language of thought. It was pointed out that this implies that the observable quantities are gauge invariant.

The second question is rarely considered in practice. Typically equations in physics are written down in order to explain certain phenomena. In this case the observable quantities have already been fixed a priori by the scope of the investigation. In the present frame work the question is tied to the question how systems specified as initial states identify themselves - e.g. as electro-magnetic fields to which Maxwell's equations of motion would apply, or as space (=space-like hyper-plane in space time) to which the equations of motion of general relativity would apply. The general answer is this. They identify themselves by properties which can be formulated in the language of thought, and which are preserved (in their totality) by the time development. Physicists call them *constraints.* Semantics demands therefore that systems with these properties are given names in natural language.

The question may arise whether all possible such properties will also occur in nature. The practical answer is that not all possible properties can be expected to give rise to phenomena that can be observed *at scales which are very large* compared to scales set by the equations. Only properties which give rise to emergent behavior can be observed in practice. This shows that fundamental physics is part of complex systems theory [9] .

In electrodynamics and Yang Mills theory, the familiar constraint is Gauss' law. It can be formulated in the present frame work in the form of statements "loop = identity". The result is shown in schematic form in figure 4.

In addition there are further properties which assert the existence of invariants. The properties of this type are linearity of the state spaces Ω_X and of the maps f between these state spaces (links b =lattice gauge fields), and the identification of the gauge group G and of Ω_X as a representation space of it. Linearity amounts to the existence of invariant operators for addition, $+ : \Omega_X \times \Omega_X \mapsto \Omega_X$ and for multiplication with numbers $* : \mathbf{C} \times \Omega_X \mapsto \Omega_X$ with associativity, commutativity and distributivity properties. [The definition of the gauge group was described before.] The defining property of invariants is that their parallel transport is path independent. As a result they can be globally defined in such a way that they commute with parallel transport.

The gauge group in Electrodynamics is Abelian. This amounts to the statement

$$s \circ s' = s' \circ s$$

for arbitrary loops $s, s' : X \mapsto X$.

The constraints in general relativity include the Gauss law. In addition there are further constraints. Their statement involves an invariant tr whose existence is also one of the constraints. It maps loops to numbers. The further constraints are shown in figures 6 The equations of motion of general relativity are shown in figure 5.

All of this discussion is at the level of formal discretizations of the standard theories.

7.2 Interpretation of the universal form of Maxwell's equations

Maxwell's equations of motion are of second order once the vector potential is introduced. Therefore the initial state needs to specify both coordinates and velocities or momenta. As a result, there will be two kind of links. They will be printed thin and fat, respectively. One could try to distinguish them by properties in an intrinsic way, e.g. by postulating that one kind has an adjoint and the other does not.

In the continuum, the canonical variables are vector potential $\mathbf{A}(x)$ and electric field $\mathbf{E}(\mathbf{x})$, and the Maxwell equations are

$$\begin{aligned} \dot{\mathbf{A}} &= -\mathbf{E} && (13) \\ \dot{\mathbf{E}} &= curl \mathbf{B} && (14) \\ \mathbf{B} &= curl \mathbf{A}. && (15) \end{aligned}$$

On the lattice one uses instead exponentiated quantities. We may label the links with target x in some way by $i = \pm 1, \pm 2, ...$ They have a direction and a length given by a vector $\hat{i}$. Choosing X and i will select a link $b_i(x)$. The opposite link to $b_i(x)$ is denoted $b_{-i}(x + \hat{i})$. Let $A_i(x)$ and $E^i(x)$ be the components of $\mathbf{A}$ and $\mathbf{E}$ in the directions of the links .

The thin links will be the parallel transporters of lattice gauge theory. Assuming a vector potential which is smooth on the scale of the lattice spacing, the parallel transporters can be approximated,

$$U(b) = P \exp\left(-\int A_\mu dx^\mu\right) \approx \exp\left(-A_i(x + \hat{i}/2)\right) \quad \text{for } b = b_i(X). \tag{16}$$

The fat links are taken to be

$$P(b) = U(b) \exp\left(-\tau E^i(x)\right) \ . \tag{17}$$

where τ is a discrete time step. $U(b)$ and $P(b)$ are complex numbers of modulus 1; the vector potential and the electric and magnetic field are regarded as pure imaginary - otherwise a factor i has to be put in the exponents. The E-variables on opposite links are related by

$$\exp\left(-\tau E^{(-i)}(x + \hat{i})\right) U(b) = U(b) \exp\left(\tau E^i(x)\right) \tag{18}$$

These quantities will be functions of a discrete time t . It follows from eqs.(13) and (18) that

$$\begin{aligned} U_{t+\tau}(b) &= U_t(b) \exp\left(-\tau E^i(x)\right) & (19) \\ &= U_t(b) P_t(-b) U_t(b) \ , & (20) \end{aligned}$$

which is the second of the universal Maxwell equations in figure 3.

Consider now the triangles Δ in the first equation in figure 3. In the limit of small lattice spacing, path ordering can be neglected and the parallel transporter around a triangle is given by the magnetic flux Φ_Δ through the triangle

$$\begin{aligned} U(\Delta) &= P \exp\left(-\int_\Delta A dx\right) = \exp(-\Phi_\Delta) & (21) \\ \Phi_\Delta &\approx B^\perp \cdot (\text{area of } \Delta) & (22) \end{aligned}$$

where $B^\perp$ is the magnetic field perpendicular to the triangle. The two triangles Δ_1 and Δ_2 on opposite sides of the link in figure 3 have opposite orientation. Therefore the factors $\exp(-\Phi_{\Delta_i})$ will cancel, except for the effect of the change of the magnetic field component $B^\perp$ in the direction perpendicular to the link and to $B^\perp$. This change is part of the component of $curl B$ in the direction of the link - all of it in 2 space dimensions. Taking the product over the pairs of triangles in all directions perpendicular to the link, one gets $\exp(-\gamma \hat{i} \cdot curl \mathbf{B_t})$ where γ is a dimensionful constant of geometric origin. The right hand side of the equation is this product multiplied with $U_t(b)$. So the equation reads

$$P_{t+\tau} = U_t \exp(-\gamma \hat{i} \cdot curl \mathbf{B_t}) \ . \tag{23}$$

For suitable choice of the time step, $\tau = \sqrt{\gamma}$, this is an exponentiated form of the Maxwell equation (14), because

$$\begin{aligned} P_{t+\tau} &= U_{t+\tau} \exp\left(-\tau E^i_{t+\tau}\right) & (24) \\ &= U_t \exp\left(\tau E^i_t\right) \exp\left(-\tau E^i_{t+\tau}\right) & (25) \\ &= U_t \exp\left(-\tau^2 \dot{E}^i_t\right) \ . & (26) \end{aligned}$$

for small τ, by definition of P and the equation of motion for U_t. This completes the discussion of the Maxwell equations.

For future reference, we rewrite eq.(21) in terms of the curvature tensor F_{ij}.

The smoothness assumption implies that the relation to continuum quantities [temporarily distinguished by greek indices] is as follows:

$$A_i(x) = \hat{i}^\mu A_\mu(x) = O(a) \tag{27}$$
$$F_{ij}(x) = \hat{i}^\mu \hat{j}^\nu F_{\mu\nu}(x) = O(a^2) \tag{28}$$

Smoothness of A requires that a suitable gauge is chosen locally.

The discretization (16) preserves the properties

$$U(-b) = U(b)^{-1} \tag{29}$$
$$U(\triangle) = \exp\left(-\frac{1}{2}F_{ij}(x) + ...\right) \tag{30}$$
$$trU(\triangle) = 1 + \frac{1}{8}trF_{ij}^2(x) + ... \tag{31}$$

for a triangle $\triangle$ with corners $x, x+\hat{i}, x+\hat{j}$. To see this one computes from the definition of the parallel transporters $U(\triangle) = 1 - \frac{1}{2}\left(\partial_i A_j - \partial_j A_i + [A_i, A_j]\right)(x) + O(a^3)$. It follows that eq.(30) holds with dots representing terms which are of order a^2 and traceless. Eq.(31) follows from this. All this is familiar from lattice gauge theory on a lattice of lattice spacing a.

The parallel transporters $U(b)$ represent the thin links b. In the case of general relativity, the vector potentials and the parallel transporters will be denoted by $\boldsymbol{\omega}_i, \mathbf{u}$ in place of A_i, U.

7.3 General relativity in discrete time

In Ashtekar's canonical formulation of general relativity in the continuum, the canonically conjugate variables (in the spinorial formulation) are as follows.

There is a vector potential $\boldsymbol{\omega}_i(x) \in sl(2, \mathbf{C})$ which governs parallel transport of spinors along paths on the space-like hyper-surface. The canonical conjugate to it is a spinorial version of the dreibein density, $2\tilde{\mathrm{e}}^i(x) \in sl(2, \mathbf{C})$. It is a traceless hermitian 2×2 matrix. It is the analog of the electric field in Yang Mills theory.

The equations and motion are

$$2\dot{\boldsymbol{\omega}}_k = \underline{N}[\mathbf{F}_{ki}, \tilde{\mathrm{e}}^i], \tag{32}$$
$$2\dot{\tilde{\mathrm{e}}}^k = \underline{N} D_j[\tilde{\mathrm{e}}^k, \tilde{\mathrm{e}}^j] \tag{33}$$

The lapse function $\underline{N}$ can be chosen arbitrarily. D_j is the $sl(2, \mathbf{C})$-covariant derivative.

The constraints are as follows. There is the Gauss constraint which is the same as in Yang Mills theory,

$$D_i \tilde{\mathrm{e}}^i = 0, \tag{34}$$

In addition the following scalar and vector constraints hold

$$H = tr\left(\mathbf{F}_{ij}\tilde{\mathrm{e}}^i\tilde{\mathrm{e}}^j\right) = 0, \tag{35}$$

$$H_j = tr\left(\mathbf{F}_{ij}\tilde{\mathrm{e}}^i\right) = 0, \tag{36}$$

$$\tag{37}$$

The second equation of motion can be rewritten with the help of the Gauss constraint as follows

$$2\dot{\tilde{\mathrm{e}}}^k = -\underline{N}[\tilde{\mathrm{e}}^j, D_j\tilde{\mathrm{e}}^k]. \tag{38}$$

This brings about the dilemma which of the two versions to choose as a candidate precursor for a universal formulation. I choose eq.(33) because it has a more natural geometric interpretation. If $\epsilon_{ijk} = \pm 1$ as usual, depending on whether ijk is an even or odd permutation of $1, 2, 3$, and if one defines

$$f = \epsilon_{ijk}[\tilde{\mathrm{e}}^j, \tilde{\mathrm{e}}^k]dx^i$$

then the right hand side of eq.(33) is expressible as a covariant total derivative of f. This parallels the situation in the Maxwell equations, where $curl B$ can also be regarded as total derivative of a 1-form $\epsilon_{ijk}B_{jk}dx^i$.

A certain combination of the Gauss and vector constraint is supposed to ensure the diffeomorphism invariance of the theory. It is a special property of space, not of the universal dynamics which is supposed to be much more generally applicable. The equations which govern the universal dynamics have a different kind of coordinate independence.

There exists in the literature a lattice formulation of general relativity in Ashtekar variables which is very similar to lattice gauge theory [21]. In the case of the 2+1-dimensional theory there is a fully consistent discrete formulation of quantum gravity due to H. Waelbroeck [20]. It uses the fact that the scalar and vector constraints in 2+1 dimensions are equivalent modulo invertibility of the dreibein to

$$F_{ij}(x) = 0.$$

In the lattice formulation, the vector potential gets replaced by parallel transporters $u(b) \in SL(2, \mathbf{C})$. Apart from the gauge group this is much as in the case of Yang Mills theory. The parallel transporters are assigned to links b of the lattice.

In the continuous time formulation, the dreibein on the lattice remains an element of the Lie algebra. It is also assigned to links of the lattice (more precisely to one of their endpoints; the dreibein transform under gauge transformations like matter fields in the adjoint representation which sit on sites x). This is very similar to the electric field in the Hamiltonian formulation of lattice gauge theory [12].

Let us again label the links $b = b_i(x)$ which enter a point x of the lattice by i. The link in opposite direction shall be $b_{-i}(x + \hat{i})$. Henceforth, labels i, j, k shall refer to links of the lattice. To avoid confusion, space indices will be labeled by $\mu, \nu, ...$ when they are needed. The basic variables are then $u(b_i(x)) \equiv u_i(x) \in SL(2, \mathbf{C})$ and $\tilde{\mathrm{e}}^i(x) \in sl(2, \mathbf{C})$

A continuous time formulation needs much a priori structure; it seems impossible to formulate it without assuming at least that our spaces Ω_X are manifolds. Because of this, time will also be discretized, and the dreibein will be exponentiated to

$$\mathcal{E}^i(x) = \exp\left(-\tau \mathrm{e}^i(x)\right) \tag{39}$$

τ will be related to the time step. As a consequence of corresponding properties of the discretized dreibein variables [20], the $\mathcal{E}$-variables on adjoint links are related by

$$u_i(x)\mathcal{E}^i(x) = \mathcal{E}^{-i}(x+\hat{\imath})u_i(x). \tag{40}$$

The Gauss law reads (to leading order in τ)

$$\prod_i \mathcal{E}^i(x) = 1$$

In the graphical representation the thin links represent parallel transporters $u(b) = u_i(x)$ and the fat links are given by

$$p(b) \equiv p_i(x) = u_i(x)\mathcal{E}^i(x) \tag{41}$$

They obey the unitarity relation $p(b^*) = p(b)^{-1}$. The $\mathcal{E}$-variables get represented as hair pins, cp. the left hand side of figure 5 (2nd equation), and the Gauss law takes again the form of figure (4).

In this approach the geometry is not in the lattice but in the parallel transporters on the links of the lattice. For simplicity I will assume that charts in the continuum which cover a sufficiently small neighborhood of a point can be represented by a regular triangular mesh, as in figure 1. In 3 space dimensions, a dense sphere packing can be used. It is assumed that the continuum vector potential is smooth on the scale of the mesh. This introduces a small parameter ϵ.

To justify the discretized version of eq.(33), the familiar relation between the group theoretical commutator $xyx^{-1}y^{-1}$ and the Lie algebra commutator $[X, Y]$ is used. It yields

$$\mathcal{E}^j(x+\hat{\jmath})\mathcal{E}^k(x+\hat{\jmath})\mathcal{E}^j(x+\hat{\jmath})^{-1}\mathcal{E}^k(x+\hat{\jmath})^{-1} = 1 + \tau^2[\tilde{\mathrm{e}}^j(x+\hat{\jmath}), \tilde{\mathrm{e}}^k(x+\hat{\jmath})] \tag{42}$$

The left hand side is represented by a composition of four hair-pins. They can be seen in figure 5 (2nd half) except that the initial and terminal thin lines are missing. The links of the triangle in the figure are $b_j(x), b_k(x+\hat{\jmath}), b_{-i}(x+\hat{\imath})$. The four hair-pins make a loop from $x+\hat{\jmath}$ to $x+\hat{\jmath}$. To be able to compose these loops for different triangles with the same side $b_i(x)$, parallel transport from $x = \hat{\jmath}$ to x is needed. There are two paths (with one or two links) to choose from. The difference does not matter because its effect is of higher order in ϵ. The choice is made so as to minimize the total number of links in the path.

The rest of the argument goes as in the case of the Maxwell equation. The contribution from triangles on opposite sides of the link $b_i(x)$ will cancel out except for a contribution proportional to the covariant divergence.

The other equation of motion and the constraints involve the curvature. There is an equation like eq.(22) which relates the parallel transporter around a triangle to the magnetic field, except that now the analog of the magnetic field is the curvature of the $SL(2,\mathbf{C})$-connection on the space-like hyper-surface.

Consider

$$u_{k,t+\tau} = u_{k,t}(1-\tau\dot{\boldsymbol{\omega}}_k + ...) \tag{43}$$

u_k is represented by the horizontal thin lines in figure 5.

The bitriangle in the first equation of figure 5 is

$$\begin{aligned}
\underline{\boldsymbol{\Delta}}_1 &= u(\triangle)^{-1}u_j(x)^* p_j(x) u(\triangle) & (44)\\
&= u(\triangle)^{-1}\left(1-\tau\tilde{\mathbf{e}}^j(x)+...\right)u(\triangle) & (45)\\
&= 1+\frac{\tau}{2}[\tilde{\mathbf{e}}^j(x),\mathbf{F}_{kj}]+... & (46)\\
&= (1-\tau\dot{\boldsymbol{\omega}}_k+...) & (47)
\end{aligned}$$

by eq.(30) and Einstein equation (32). Comparing, we see that the figure reproduces the time evolution (43).

8 The Dirac equation

The content of this section is based on joint work with B. Holm and D. Lübbert [19].

The massless Dirac or Weyl equation in flat 4-dimensional space time reads

$$-i\hbar\dot{\psi} = i\hbar c\boldsymbol{\alpha}^i\cdot\partial_i\psi \equiv \hbar c\mathbf{D}\psi \tag{48}$$

with matrices α^i which obey the standard anti-commutation relations

$$\{\boldsymbol{\alpha}^i,\boldsymbol{\alpha}^j\}_+ = 2\delta^{ij}\mathbf{1}\ (i,j=1...3) \tag{49}$$

It follows from these relations that $\mathbf{D}$ is a square root of the negative Laplacian $-\Delta$.

The appropriate inverse dreibein for a flat space is $e_a^{\ i} = \delta_a^{\ i}$ with $e^{-1} = det(e_a^{\ i}) = 1$. It follows that the Ashtekar dreibein-variable in flat space

$$\boldsymbol{\alpha}^i = \tilde{\mathbf{e}}^i \equiv e e_a^{\ j}\boldsymbol{\sigma}^a \tag{50}$$

obeys the anti-commutation relations (49).

Let us denote by $-i$ the opposite direction to i; $\partial_{-i} = -\partial_i$. There is no distinct positive direction, and $\boldsymbol{\alpha}^i\partial_i$ should have a meaning independent of choices of positive directions. Therefore one should set

$$\boldsymbol{\alpha}^i = -\boldsymbol{\alpha}^{-i}\ . \tag{51}$$

The anti-commutation relations can now be written as

$$\begin{aligned}
\boldsymbol{\alpha}^i\boldsymbol{\alpha}^j &= -\boldsymbol{\alpha}^j\boldsymbol{\alpha}^i\quad (i\neq\pm j)\ , & (52)\\
\boldsymbol{\alpha}^i\boldsymbol{\alpha}^{-i} &= -1\ , & (53)
\end{aligned}$$

and $\boldsymbol{\alpha}^{2i} \equiv \boldsymbol{\alpha}^i \boldsymbol{\alpha}^i = 1$. We see here -signs which are of crucial importance, especially the second one. The product is a product of matrices.

In the Kogut Susskind discretization of the Dirac equation, a cubic lattice with lattice sites x substitutes for continuous 3-space. In place of the Dirac $\boldsymbol{\alpha}$-matrices one has numbers $\eta^i(x)$ attached to links $(x+\hat{i}, x)$ between nearest neighbors x and $x+\hat{i}$ on the lattice ($\hat{i}$ is the lattice vector in i-direction). In place of the Dirac algebra (52) one has the relations

$$\eta^i(x+j)\eta^j(x) = -\eta^j(x+i)\eta^i(x), \ (i \neq \pm j) \tag{54}$$
$$\eta^{-i}(x+\hat{i})\eta^i(x) = -1. \tag{55}$$

Usually one requires in addition

$$\eta^{2i}(x) \equiv \eta^i(x+\hat{i})\eta^i(x) = 1. \tag{56}$$

But this relation can be abandoned. The η-s sit on links. We may think of them as parallel transporters of an (external) gauge field. (In the Kogut Susskind formalism the gauge group is $\mathbf{Z}_2$.) Validity of eq(56) can then be assured by a gauge transformation. This is seen as follows.

It follows from equations (54) that

$$\eta^{2i}(x+2j)\eta^{2j}(x) = \eta^{2j}(x+2i)\eta^{2i}(x) \tag{57}$$

for all i, j. We may restrict attention to a sub-lattice of twice the lattice spacing, with parallel transporters η^{2i}. Eq.(57) tells us that η^{2i} is a pure gauge. Therefore it can be gauged away by a gauge transformation

$$\psi(x) \mapsto \gamma(x)\psi(x), \ \gamma(x) \in \mathbf{C}.$$

If the $\eta' s$ are either numbers ± 1 or $\pm i$ then $\gamma(x)$ can be chosen in $\mathbf{Z}_2$.

The Kogut Susskind discretization of the massless Dirac equation in continuous time is the $\delta t \mapsto 0$ limit of the following equation

$$\psi(x, t+\delta t) = \psi(x,t) + \delta t \sum_i \eta^{-i}(x+\hat{i})\psi(x+\hat{i}). \tag{58}$$

The sum goes over all nearest neighbors $x+\hat{i}$ of x, i.e. over positive and negative directions. $\hbar$ dropped out and we put $c=1$. The equation is invariant under the aforementioned gauge transformations. One may demand $\eta^i(x) = \pm 1$. In this case the gauge freedom restricts to $\mathbf{Z}(2)$-gauge transformations $\gamma(x) = \pm 1$. The group $\mathbf{Z}(2)$ is the center of $SL(2, \mathbf{C})$. It remains as a gauge group after one uses the flatness condition to gauge away the $SO(3,1)$-connection.

So far we considered a cubic lattice. The consideration can be generalized to a triangular lattice (dense sphere packing) [19]. It turns out that one needs to augment the constraints (54) by the additional condition that the square of the parallel transporter around a triangle is (-1) in order to deduce the unfrustratedness condition 57. As a result the η's take values $\pm i$.

Now we wish to embed these formulae into the general frame work. In the case of the simple Dirac equation in flat space and without gauge fields, the parallel transporter along the link from x to $x+\hat{i}$ shall be given by the number $\eta^i(x)$ and Ω_x is isomorphic to $\mathbf{C}$.

As discussed before, we wish to think of the spinor fields $\psi(x)$ and their adjoints $\bar{\psi}(x)$ as links to and from ∞. We parameterize these links by elements, also denoted $\psi(x)$ of the linear space Ω_x and $\bar{\psi}(x)$ in its dual space, respectively. We use the addition $+$ in these vector spaces in order to fix the composition law $\underset{\infty}{\circ}$ of arrows at ∞ in a manner which will now be described.

We wish to exhibit properties of the initial state which guarantee that the universal dynamics reduces to the Dirac equation. The composition law is determined by the initial conditions. Therefore we are free to select an appropriate composition law, and to use other properties of the initial state to construct it.

The composition law $\underset{X}{\circ}$ of arrows at finite points X of space is decreed to be given by the composition of maps between spaces Ω, in accordance with the representation theorem. This theorem would tell us that there is also a space Ω_∞. But this is of no help, because this space could only be constructed once we know the composition law $\underset{\infty}{\circ}$. In accordance with the discussion in section (2) we construct this composition by imposing relations between paths.

A general path which goes from X to Y, meeting ∞ a number $N>0$ of times will have the form

$$...\psi \underset{\infty}{\circ} l_1 \underset{\infty}{\circ} ...l_{N-1} \underset{\infty}{\circ} \bar{\psi}...$$

where $\psi : \infty \mapsto X$, and $\bar{\psi}(x) : Y \mapsto \infty$ are links, and $l_i : \infty \mapsto \infty$ are paths. The dots represent arrows which come from paths that do not touch ∞. We impose the following relation. If l_1 is a triangle of the form

$$\psi^* \circ u_1 \circ \zeta_1 : \infty \mapsto \infty \tag{59}$$

where u_1 is a link between finite points, then

$$\psi \circ l_1 = \psi + u_1 \circ \zeta_1 \ . \tag{60}$$

while $\psi \circ l_1 = 0$ otherwise. If $l_1...l_M$ are of the form (59) with different u_i and ζ_i then the relation may be applied several times to produce a sum.

$$\psi \circ l_1 \circ ... \circ l_M = \psi + \sum u_i \circ \zeta_i \ .$$

Using this we see that the Kogut Susskind discretization of the Dirac equation can be written in the form of figure 2.

If there is a gauge field, the Ω_x become representation spaces for the gauge group, and the η^i get multiplied with parallel transporters that are furnished by the gauge field. Only the first of the relations (54) survives this, and the square of the covariant Dirac operator is no longer equal to the negative Laplacian.

In curved space, the parallel transporters are associated with the Ashtekar variables ω_i and the η^i involve the Ashtekar variables $\tilde{\mathbf{e}}^i$.

9 Practical improvements

If one wants to use discretized versions of the Maxwell Yang Mills or Einstein Ashtekar equations in computer simulations, some improvement is called for. First, ons should *include growth processes in the dynamics* such that the grid gets refined when the electric or magnetic field (or their analogue) get large. Secondly, one ought to guard against accumulating violations of the constraints from rounding or discretization errors by including suitable gradient terms which tend to restore the constraints.

10 Quantum Systems and Quantum Space Time

Now we wish to proceed to a quantized theory. We use Schroedinger wave functions. In standard quantum mechanics, they depend either on coordinates. or on momenta, but not both. Accordingly we assign complex amplitudes

$$\Psi(S) \in \mathbf{C} \tag{61}$$

to systems S which contain only one kind of links. If we come from a classical description, we must choose either thin links b or fat links $\mathbf{b}$. In gravity we settle for the first possibility; the thin links represent $SL(2\mathbf{C})$ parallel transporters in this case.

Before going into gravity, let us consider a simpler example.

Example Quantum mechanical motion of particles in space *can be described by a universal dynamics. We may picture a system of objects ("space points") linked by bidirectional arrows plus additional objects ("point particles") linked to space points (their positions) by one unidirectional arrow each, as in figure 11. The space is considered constant, the particles may move as indicated in figure 11. In the case of a single particle , the system $S = S[x]$ is then determined by specifying the position x of the particle and we may identify our Schrödinger amplitude $\Psi(S[x])$ with a standard 1-particle Schrödinger wave function $\Psi(x)$.*

A Schrödinger equation for the complex amplitude of such a system S reads

$$i\dot{\Psi}(S) = -\sum_{\mu}[\Psi(\mu S) - \Psi(S)] \ . \tag{62}$$

Summation is over moves μ of individual particles as in figure 11. For a single particle of mass m on a triangular or cubic lattice of space points, this is the standard discretization of the Schrödinger equation for free motion. To see this, recall the standard discretization of the Laplace operator on a cubic lattice of lattice spacing $a = 1$,

$$\Delta\Psi(x) = \sum_{\mu}[\Psi(x+\hat{\mu}) - \Psi(x)]$$

where $x + \hat{\mu}$ is the nearest neighbour of x on the lattice in μ direction. Units of time $\hbar/2m$ are set to 1.

At least in the case of a compact space, the dynamics of quantum gravity is the most universal one that one can conceive of. There is no time dependence because the Hamiltonian is zero. The interpretation of this fact is being debated. It is tempting to interpret Parmenides as saying that dynamics comes into the world only through the separation between observer and observed. I come back to this, but for now let us accept that the only remaining problem in quantum gravity is to find the space of wave functions. And for any system is a nontrivial problem to find this space because of the constraints.

The wave functions will be required to be gauge invariant. More generally, the wave function cannot assign different amplitudes to systems S which cannot be distinguished in an intrinsic way by properties which can be formulated in the language of thought.

In the quantum theory, gauge invariant functions of the parallel transporters will become multiplication operators. Examples are operators $tr\, l$, where $l = b_1 \circ ... \circ b_n$ are loops and where the trace tr is a a gauge invariant function on loops. [9] Operators which involve canonical momenta ("fat links") in the classical case will have to be represented as substitution operators. They are called "grasp operators" in quantum gravity [3]. They map to linear combinations of wave functions for values of the argument S which are obtained from each other by some operation on S which is specified by the observable.

The interpretation of this prescription is somewhat subtle though. How would one specify a special loop l in a general system S? In general this is impossible. But there are observables like $\prod tr\triangle$ (products over all triangular loops in S) which are well defined products of such loops for all systems. Operators which have a physical meaning, such as the constraints, will have to be of this type.

This gives us one immediate candidate for a solution to all conceivable constraints of the kind which demands invariance of the wave function under a particular possible dynamics. If the operation of the substitution operators on S is compounded from local actions, they can not change the category of S. [This generalizes the classical result of section 4.1.] Therefore wave functions $\Psi(S)$ which depend on S only through its category will obey all reasonable constraints. This reminds of the construction of thermodynamic ensembles from conserved quantities.

Let us specialize to quantum gravity. How can one label a basis in the space of wave functions Ψ? The problem has two parts. A system S is specified by a graph $G = Graph(S)$ and by relations between the paths on G. In the traditional formulation, these relations are given implicitly by prescribing a gauge field configuration on G. The Mandelstam relations are among these [25]. The first problem then is to label wave functions $\psi_{G,p}$ whose arguments are gauge field configurations, i.e. systems S with a given graph G. The second part of the problem concerns the generalization to wave functions which are defined for

[9] Such a function tr always exists. An example is as follows. Consider loops $l : X \mapsto X$ and set $tr\, l = 1$ if $l = \iota_X$ and 0 otherwise. When Ω_X are linear spaces, the trace of linear operators can be used instead. Remember that the existence of invariants like tr is to be read off from the initial state

arbitrary systems S. I will restrict attention to wave functions whose support consists of systems whose graph is the skeleton of a simplicial complex.

The first problem has recently been solved within the loop space approach to quantum gravity [21, 22] with the help of Penrose spin networks [13, 25, 24]. The issue had been to take account of the relations between traces of loops which hold true for every $SL(2, \mathbf{C})$ gauge field.

A Penrose spin network P is a graph with an assignment of a positive integer $n(b)$, called its color, to every undirected link b of the graph, subject to certain conditions. If all the nodes are trivalent, the condition is as follows. The sum of the colors of the three links incident on a node has to be even, and none of them is larger than the sum of the other two. Since I prefer to work with directed graphs where every link b comes with a link b^* in the opposite direction, I set

$$n(b) = n(b^*).$$

An embedding $p = \gamma_G(P)$ of a Penrose spin network in a graph G is a map of distinct nodes of P to distinct nodes of G and of links of P between nodes to paths in G between corresponding nodes. The paths must not intersect except as prescribed at the endpoints.

To be in agreement with the literature, I change nomenclature. What has been called a loop up to now will henceforth be called a simple loop. And loops α are collections of simple loops $\alpha_1 ... \alpha_k$. One sets

$$tr\, \alpha = (-)^k \prod_{a=1}^{k} tr\, \alpha_a$$

To every embedded Penrose spin network p, a formal sum of loops α with coefficients $c_\alpha = \pm 1$ is assigned by a certain prescription

$$p \Rightarrow \sum_\alpha c_\alpha \alpha$$

Let $l(b)$ be the number of times the path passes through link b of G. The embedded spin network specifies an integer $n(b) \geq 0$ for links of G. The prescription is such that $n(b) = l(b) + l(b^*)$ and there is antisymmetry under operations of reconnecting the loops by permuting the end points of instances of the same link b. For details, the reader is referred to the literature. The basis of states is given by

$$\Psi_{G,p}(S) = \sum c_\alpha tr \alpha \tag{63}$$

Given the graph G, one obtains in this way a linearly independent set of gauge invariant functions of systems S with given graph G.

The action of loop operators on such states has been defined in the literature for loops with arbitrary number of "fat links" (dreibeins). In the above discussion of classical gravity, an exponentiated dreibein was used, because we had to mimic addition by use of the composition $\circ$. In quantum gravity, there is no need for this, because the linearity of the state space supplies an operator $+$. The Lie algebra valued dreibein can be used.

We may adopt the construction, but there remains the above mentioned second problem to be faced. We do not wish to consider the space as preexisting (e.g. as a manifold) with only a geometry that remains to to be put on it. Instead we want to define wave functions $\Psi_p : S \mapsto \mathbf{C}$ on arbitrary systems S. There is no way to formulate an intrinsic specification of an embedded Penrose spin network which makes sense for a completely arbitrary system S. But there are distinguished classes of such embedded spin networks and we may sum or average av over the representatives. Let me write av for this sum or average. If there is no representative for a given system S, the result is zero.

The solution of the diffeomorphism constraint in quantum gravity through the use of knots [23, 21] and its elaboration in terms of Penrose spin-networks [25] suggests what to do. The knot class K of an embedded spin network is an equivalence class of spin networks which share an intrinsic property. It is called an s-knot. Embedded networks which are obtained from each other by homotopic deformation of the paths represent the same s-knot.

This leads us to try

$$\Psi_K(S) = N_K \, av_{p \in K} \, \Psi_{Graph(S),p}(S) \tag{64}$$

where N_K is an arbitrary normalization factor. These states are automatically diffeomorphism invariant.

I wish to suggest a variation on this theme. Let us consider systems with a distinguished object O, called its root. The idea is that all the states are supposed to be subject to examination by one and the same observer which is somewhere and thus marks a point in space. Briefly, call O the observer. Loops from O to an object X and back may be interpreted as queries from O to X and answer back from X to O. The message can be influenced by the medium through which it passes. The construction above aims at labeling states by quantum numbers which record observable properties. If the observation is to be made by O, also the spin network which is to be embedded should have a distinguished node, and this node is to be mapped on O. This leads to "based knots". Parmenides and the idea of considering a quantum system together with its observer suggest, moreover, to require gauge invariance only under gauge transformations which are trivial at O and to admit functions tr which are invariant in this restricted sense only and which may depend on the observer. This enlarges the space of observables.

Systems in quantum mechanics which include an observer are not usually considered in text books. But they play a role in recent experiments on quantum erasers [27]. These experiments show that the effect of a measurement on a quantum system can be erased again, under certain conditions.

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Appendix A: Proof of the main representation theorem

A slightly more elaborate version of the representation theorem was proven by the author in [1]. For the convenience of the reader, the statement and proof of this theorem is reproduced here, and it is shown how the version in the main text, which is closer to lattice gauge theory, is obtained as a corollary.

I use a slightly more elaborate notation than in the main text. Given the system K with objects $X, Y, ...$, denote by $Mor(Y, X)$ the set of arrows $f : X \mapsto Y$ and $g \underset{Y}{\circ} f : X \mapsto Z$ for the composite of $f \in Mor(Y, X)$ with $g \in Mor(Z, Y)$.

Representation theorem 1 (Representation of a system as a communication network) *Every system K permits a faithful representation with the following properties*

To every object X there exists an input space A_X and an output space Ω_X. The input space contains a distinguished element $\emptyset$ ("empty input"). Arrows $f \in Mor(Y, X), g \in Mor(Z, Y)$ and objects X act as maps

$$X : A_X \mapsto \Omega_X, \tag{65}$$

$$\iota_X : \Omega_X \mapsto A_X \tag{66}$$

$$f : \Omega_X \mapsto A_Y \tag{67}$$

with the properties

$$X\iota_X = id : \Omega_X \mapsto \Omega_X \quad , \quad \iota_X X = id : A_X \mapsto A_X \ , \tag{68}$$

$$g \underset{Y}{\circ} f = gYf : \Omega_X \mapsto A_Z \ . \tag{69}$$

It should be noted that ι_X does not act as the identity map in general in this context.

Given this version of the representation theorem, we restrict attention to the output spaces Ω_X and to maps $\hat{f} = Y \circ f : \Omega_X \mapsto \Omega_Y$. Renaming $\hat{f}$ into f we obtain the representation theorem of the main text.

Proof of the representation theorem 1 for categories

Given a system K, we write $Mor(Y, *)$ for the set of all its arrows to Y etc.. We define

$$In(Y) = Mor(Y, *) \ , \ Out(Y) = Mor(*, Y) \ .$$

We write $X = \alpha(f)$ if $f \in Mor(Y, X) \subset In(Y)$, and correspondingly $Z = \omega(f)$ if $f \in Mor(Z, Y) \subset Out(Y)$. The output space will be defined as a subspace

Ω_Y of Ω_Y^{virt}. Ω_Y^{virt} consists of maps

$$\zeta : Out_Y \mapsto Mor(*,*)$$

with the property $\zeta(f) \in Mor_K(\omega(f),*)$.

An object Y will act as a map

$$Y : In(Y) \mapsto \Omega_Y .$$

according to

$$Yf(g) = g \underset{Y}{\circ} f \quad (g \in Out(Y)).$$

The output space is defined as the image of Y, and the input space as space of equivalence classes (if necessary) of elements of $In_K(Y)$, which Y maps into the same $\zeta \in \Omega_Y^{virt}$.

$$\Omega_Y = IM\ Y \subset \Omega_Y^{virt} , \tag{70}$$
$$A_Y = In(Y)/KER\ Y . \tag{71}$$

Y is invertible as a map from A_Y to Ω_Y. Its inverse is ι_Y. The empty input $\emptyset \in A_Y$ is defined as the equivalence class of $\iota_Y \in Mor(Y,Y) \subset In(Y)$.

An arrow $f \in Mor(Y,X)$ is defined as a map $\Omega_X \mapsto A_Y$ by use of the map $\iota_X : \Omega_Y \mapsto A_Y$, as follows.

$$f = \hat{f} \underset{X}{\circ} \iota_X , \tag{72}$$
$$\hat{f}(g) = f \underset{X}{\circ} g \quad \text{for } g \in Mor(X,*) . \tag{73}$$

The last formula defines $\hat{f}$ as a map from $In(X)$ to $In(Y)$. This map passes to equivalence classes (71) thereby defining a map $A_X \mapsto A_Y$, The composition rule (69) holds.

Appendix B: General Relativity in Ashtekar variables

For the convenience of the reader I will briefly review the canonical formalism for general relativity in Ashtekar variables.

Before dealing with Ashtekar variables, let me briefly examine classical general relativity in order to see what a priori structural assumptions are made.

In classical general relativity one deals with a 4-dimensional space time manifold $\mathcal{M}$ and with a dynamically determined geometry on $\mathcal{M}$. The geometry provides a connection in the tangent bundle which is compatible with a Lorentzian metric. Field equations for the metric and the connection are derived from a variational principle. The vanishing of the torsion is one of these field equations.

In the vierbein formalism, the connection in the tangent bundle can be thought to be constructed in two steps.

1. There is a connection in a vector bundle over $\mathcal{M}$ with fibers $V_x \approx V^{(\frac{1}{2},\frac{1}{2})} \approx \mathbf{R}^4$. This connection preserves a bilinear form $<,>_x$ of signature $(+---)$ on the fibers.

2. The fibers V_x are identified with the tangent spaces $T_x\mathcal{M}$.

$V^{(\frac{1}{2},\frac{1}{2})}$ is the 4-dimensional real representation space of the Lorentz group ("vector representation").

The connection specifies parallel transporters. They are linear maps which preserve the bilinear form

$$\begin{aligned} \mathcal{U}(C) \quad &: \quad V_x \mapsto V_y, & (74)\\ <\mathcal{U}(C)v, \mathcal{U}(C)w>_y \quad &= \quad <v,w>_x & (75)\end{aligned}$$

They transport vectors $v, w \in V_x$ along piecewise smooth paths C on from x to y.

The identification is provided by a vierbein. It specifies an invertible map from the tangent space to the internal space [10]

$$\mathcal{E}(x) : T_x\mathcal{M} \mapsto V_x \tag{76}$$

for every x. By virtue of the identification, the bilinear form $<,>_x$ on the fibers becomes a Lorentz metric g on $\mathcal{M}$, viz. $g(X,Y) = <\mathcal{E}(x)X, \mathcal{E}(y)Y>_x$ for $X, Y \in T_x\mathcal{M}$.

In this manner, general relativity appears as a gauge theory with gauge group isomorphic to the Lorentz group $SO(3,1)$ and with a distinguished field, the vierbein field. The action has a particular form.

The standard description is obtained by introducing coordinate systems on charts of $\mathcal{M}$ and a moving frame on each chart. The moving frame specifies a pseudo-orthonormal basis $\mathbf{f}(x) = (f_0(x), f_1(x), f_2(x), f_3(x))$ of V_x for every x in the chart. Pseudo-orthogonality reads

$$< f_\alpha(x), f_\beta(x) >_x = \eta_{\alpha\beta}$$

with $\eta_{\alpha\beta} = diag(+1,-1,-1,-1)$. The moving frame serves to convert linear maps into matrices.

The pseudo-orthonormal frames $\mathbf{f}(x)$ form the fibers of a principal fiber bundle whose structure group is the Lorentz group $SO(3,1)$. Parallel transport of vectors induces parallel transport of frames and thereby a connection on a principal fiber bundle.

The coordinate system specifies a basis ∂_μ in the tangent spaces $T_x\mathcal{M}$. Expanding everything in sight, one gets the components of the vierbein and of its inverse , the components of the metric tensor, and the parallel transport matrices $\mathbf{U}(C) \in SO(3,1)$ with entries $U^\alpha_{\ \beta}$.

$$\mathcal{E}(x)\partial_\mu \quad = \quad E_\mu^{\ \alpha}(x) f_\alpha(x), \tag{77}$$

[10] Beware of confusion. This is not the same $\mathcal{E}$ as in the main text.

$$\mathcal{E}(x)^{-1} f_\alpha(x) = E_\alpha{}^\mu(x)\partial_\mu, \tag{78}$$
$$E_\mu{}^\alpha \eta_{\alpha\beta} E_\nu{}^\beta = g_{\mu\nu}(x) \tag{79}$$
$$\mathcal{U}(C) f_\alpha(x) = f_\beta(y) U^\beta{}_\alpha(C). \tag{80}$$

The moving frame serves to convert linear maps into matrices. The parallel transport matrix $\mathbf{U}(C)$ for infinitesimal paths C from a point x with coordinates x^μ to a neighboring point with coordinates $x^\mu + \delta x^\mu$ defines the vector potential $\Gamma_\mu(x) = (\Gamma^\alpha{}_{\beta\mu}(x))$

$$\mathbf{U}(C) = 1 - \Gamma_\mu(x)\delta x^\mu. \tag{81}$$

The entries of the vector potential are also known as the connection coefficients in the anholonomic basis provided by f_α.

The matrix $\mathbf{R}_{\mu\nu}$ whose entries are the anholonomic components $R^\alpha{}_{\beta\mu\nu}$ of the field strength- or curvature-tensor $\mathcal{F}_{\mu\nu}(x) : V_x \mapsto V_y$ are given by the standard formula

$$\mathcal{F}_{\mu\nu}(x) f_\alpha(x) = f_\beta(x) R^\beta{}_{\alpha\mu\nu}(x), \tag{82}$$
$$\mathbf{R}_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu. \tag{83}$$

One may compute the parallel transporter $\mathcal{U}(C)$ around an infinitesimal triangle $C = \triangle$ whose corners have coordinates $\{x^\mu\}, \{x^\mu + \delta y^\mu\}, \{x^\mu + \delta z^\mu\}$. The result can be stated in basis independent form as

$$\mathcal{U}(\triangle) = 1 - \frac{1}{2}\mathcal{F}_{\mu\nu}(x)\delta y^\mu \delta z^\nu + \dots . \tag{84}$$

The differentials $\delta y^\mu, \delta z^\nu$ should be regarded as anti-commuting.

Gauge transformations are determines by matrices $S(x) = (S^\alpha{}_\beta(x)) \in SO(3,1)$. A (passive) gauge transformation is a change of moving frame

$$f_\alpha(x) \mapsto f_\beta(x) S^\beta{}_\alpha(x). \tag{85}$$

This transformation preserves pseudo-orthonormality. The parallel transport matrix, vierbein components and vector potential transform in the familiar way under such gauge transformations.

Connections in spinor space. Ashtekar variables

The Ashtekar variables appear very naturally if one starts from parallel transport of spinors rather than 4-vectors. Such parallel transport of spinors must be considered anyway when one wants to describe matter by wave functions for spin $\frac{1}{2}$ particles. The gauge group is then the quantum mechanical Lorentz group $SL(2, \mathbf{C})$.

Because of the structural assumptions of the standard theory, the parallel transport of vectors in an arbitrary representation space of the structure group determines the parallel transport of vectors in any representation space.

The fibers $V_x^+ \approx V^{(\frac{1}{2},0)} \approx \mathbf{C}^2$ and $V_x^- \approx V^{(0,\frac{1}{2})} \approx \mathbf{C}^2$ are now isomorphic to 2-dimensional complex representation spaces of $SL(2,\mathbf{C})$. One has

$$V^{(\frac{1}{2},\frac{1}{2})} = V^{(\frac{1}{2},0)} \otimes V^{(0,\frac{1}{2})}. \tag{86}$$

More precisely, $V^{(\frac{1}{2},\frac{1}{2})}$ is a real subspace of the complex representation space $V^{(\frac{1}{2},0)} \otimes V^{(0,\frac{1}{2})}$. This identification can serve to construct a moving frame in $V^{(\frac{1}{2},0)} \otimes V^{(0,\frac{1}{2})}$ from a moving frame in $V^{(\frac{1}{2},\frac{1}{2})}$. The parallel transport matrices $\mathbf{u}(C) \in SL(2,\mathbf{C})$ for vectors in $V_x^+ \approx V^{(\frac{1}{2},0)}$ and $U(C) \in SO(3,1)$ for vectors in $V_x \approx V^{(\frac{1}{2},\frac{1}{2})}$ are related by the fundamental formula of spinor calculus,

$$\begin{aligned} A\boldsymbol{\sigma}_\alpha A^* &= \boldsymbol{\sigma}_\beta \Lambda^\beta{}_\alpha(A) \quad \text{for } \mathbf{A} \in SL(2,\mathbf{C}), && (87)\\ \mathbf{X}\boldsymbol{\sigma}_\alpha + \boldsymbol{\sigma}_\alpha \mathbf{X}^* &= \boldsymbol{\sigma}_\beta \Lambda^\beta{}_\alpha(\mathbf{X}) \quad \text{for } \mathbf{X} \in sl(2,\mathbf{C}). && (88) \end{aligned}$$

$\boldsymbol{\sigma}_i$ are the Pauli matrices for $i = 1,2,3$ and $\boldsymbol{\sigma}_0 = \mathbf{1}$ (2×2 identity matrix). This formula yields the Lorentz transformation $\Lambda(\mathbf{A})$ which is associated with $\mathbf{A} \in SL(2,\mathbf{C})$, and similarly for elements of the Lie algebra $sl(2,\mathbf{C})$.

We will use boldface letters to characterize complex 2×2 matrices throughout.

The parallel transport matrices $\mathbf{u}(C)$, the vector potential $\boldsymbol{\omega}_\mu \in sl(2,\mathbf{C})$ and the field strength matrix $\mathbf{F}_{\mu\nu} \in sl(2,\mathbf{C})$ in spinor space obey the following relations

$$\begin{aligned} \mathbf{u}(C)\boldsymbol{\sigma}_\alpha \mathbf{u}(C)^* &= \boldsymbol{\sigma}_\beta U(C)^\beta{}_\alpha, && (89)\\ \mathbf{F}_{\mu\nu}(x)\boldsymbol{\sigma}_\alpha + \boldsymbol{\sigma}_\alpha \mathbf{F}_{\mu\nu}(x)^* &= \boldsymbol{\sigma}_\beta R^\beta{}_{\alpha\mu\nu}(x) && (90)\\ \mathbf{u}(C) &= 1 - \boldsymbol{\omega}_\mu(x)\delta x^\mu + ... && (91)\\ \mathbf{F}_{\mu\nu} &= \partial_\mu \boldsymbol{\omega}_\nu - \partial_\nu \boldsymbol{\omega}_\mu + \boldsymbol{\omega}_\mu \boldsymbol{\omega}_\nu - \boldsymbol{\omega}_\nu \boldsymbol{\omega}_\mu. && (92) \end{aligned}$$

One converts also the vierbein $\mathcal{E}$ to a spinorial basis. For a fixed μ, the vector $\mathcal{E}(x)\partial_\mu \in V_x$. Because of the identification (86) of representation spaces, we may also regard $\mathcal{E}(x)\partial_\mu$ as an element of $V_x^+ \otimes V_x^-$.

To obtain convenient formulas, one introduces $\tilde{\boldsymbol{\sigma}}_i = -\boldsymbol{\sigma}_i$ ($i = 1,2,3$) and $\tilde{\boldsymbol{\sigma}}_0 = \boldsymbol{\sigma}_0$ so that

$$\begin{aligned} \boldsymbol{\sigma}_\mu \tilde{\boldsymbol{\sigma}}_\nu + \boldsymbol{\sigma}_\nu \tilde{\boldsymbol{\sigma}}_\mu &= 2\eta_{\mu\nu}, && (93)\\ \epsilon\, {}^t\boldsymbol{\sigma}_\mu \epsilon^{-1} &= \tilde{\boldsymbol{\sigma}}_\mu. && (94) \end{aligned}$$

where ϵ is the completely antisymmetric tensor in two dimensions, and ${}^t\boldsymbol{\sigma}$ is the transpose of the matrix $\boldsymbol{\sigma}$.

We introduce two (hermitian) 2×2-matrices $\mathbf{E}^\mu$ and

$$\tilde{\mathbf{E}}^\mu = \epsilon\, {}^t\mathbf{E}^\mu \epsilon^{-1}$$

as spinorial versions of the inverse vierbein $(E_\alpha{}^\mu)$.

$$\begin{aligned} E_\alpha{}^\mu &= tr\left(\boldsymbol{\sigma}_\alpha \tilde{\mathbf{E}}^\mu\right) = tr\left(\tilde{\boldsymbol{\sigma}}_\alpha \mathbf{E}^\mu\right), && (95)\\ \boldsymbol{\sigma}_\beta E^{\beta\mu} &= \mathbf{E}^\mu. && (96) \end{aligned}$$

The Einstein action will involve the volume form $E(x)d^4x$ with

$$E = det(E_\mu{}^\alpha) = det(E_\alpha{}^\mu)^{-1} = \sqrt{(}-g).$$

Using the relation (90) of the field strength and the definitions of the spinorial versions of the vierbein one finds

$$\begin{aligned} \text{Einstein action} &= \int d^4x L; && (97)\\ L &= EE_\alpha{}^\mu E^{\beta\nu} R^\alpha{}_{\beta\mu\nu} && (98)\\ &= E\ tr\left\{\mathbf{F}_{\mu\nu}\mathbf{E}^\nu \tilde{\mathbf{E}}^\mu + \mathbf{E}^{*\nu}\mathbf{F}^*_{\mu\nu}\tilde{\mathbf{E}}^{*\mu}\right\}. && (99)\end{aligned}$$

By a choice of gauge, i.e. of a suitable moving frame, the inverse vierbein is brought to the form

$$\begin{pmatrix} N^{-1} & -N^{-1}N^m \\ 0 & e_a{}^i \end{pmatrix} \tag{100}$$

It follows that

$$E = Ne, \quad e = det(e_a{}^i)^{-1}. \tag{101}$$

N and N^m are known as lapse and shift functions, and $(\mathbf{e}_a{}^i)$ is the inverse dreibein, with holonomic indices $i = 1, 2, 3$ and anholonomic indices $a = 1, 2, 3$. Its spinorial version $\mathbf{e}^i$ is given by

$$\begin{aligned} e_a{}^i &= \frac{1}{2} tr\left(\tilde{\boldsymbol{\sigma}}_a \mathbf{e}^i\right), && (102)\\ \mathbf{E}^i &= \mathbf{e}^i + E_0{}^i \boldsymbol{\sigma}_0. && (103)\end{aligned}$$

Finally one introduces

$$\begin{aligned} \underline{N} &= Ne^{-1} && (104)\\ \tilde{\mathbf{e}}^i &= e\mathbf{e}^i. && (105)\end{aligned}$$

Consider now a space-like surface $x^0 = t$. The parallel transport along curves C within the surface is determined by the space components of $\boldsymbol{\omega}_i$ of the vector potential.

The Ashtekar variables are the 2×2 matrices $\boldsymbol{\omega}_i$ and $\tilde{\mathbf{e}}^i$.

Canonical formalism

Dirac's canonical formalism with constraints is used to bring the equation of motion to Hamiltonian form and to determine constraints on the initial data. There is a subtle point, however. Ashtekar applies the canonical formalism to a theory with a gauge group $SL(2, \mathbf{C}) \times SL(2, \mathbf{C})$ ("complex relativity"). This means that left handed and right handed spinors have *independent* parallel transporters, with independent vector potentials $\boldsymbol{\omega}$ and $\boldsymbol{\omega}^*$, and there are also two independent vierbein variables E and E^*. This is quite natural from the point of view of the philosophy of this paper. To compare the left and right handed spinors, one would need to assume that there exists an invariant operation, complex conjugation, which commutes with parallel transport. It is natural to consider the existence of such an invariant and an associated relation between parallel transporters as a distinguishing feature of initial states in general relativity, but not as *a priori* structure.

Dirac's formalism is described in detail in the text book [29]. In the course of the analysis, first class and second class constraints appear. The second class constraints must be imposed as strong constraints, i.e. they are equalities on all the phase space.

As a warm up exercise the reader may consider the following nonstandard form of the Maxwell action

$$L_{Maxwell} = -\int d^4x \left\{ (\partial_\mu A_\nu - \partial_\nu A_\mu) F^{\mu\nu} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \right\} \tag{106}$$

The condition $F_{ij} = \partial_i A_j - \partial_j A_i$ arises in the form of a second class constraint. The standard Hamiltonian is obtained and the Gauss law is a first class constraint.

Let us return to the Einstein action. $\boldsymbol{\omega}_\mu$ are complex traceless 2×2 matrices. Let me repeat: Ashtekar deviates from the rules in the book in an important way. He starts with the assumption that the $sl(2,\mathbf{C})$ vector potentials $\boldsymbol{\omega}$ and $\boldsymbol{\omega}^*$ are independent variables to begin with, and also the dreibein variables which come to multiply $\mathbf{F}_{\mu\nu}$ and $\mathbf{F}^*_{\mu\nu}$ in the formula for the Einstein action are independent variables. In other words, he starts with complex general relativity; the reality constraints are only imposed to select among the solutions.

Proceeding in this way, attention is restricted to the first term in the Einstein action (99) which involves $\boldsymbol{\omega}_\mu$. The first step is the determination of the conjugate variables $\boldsymbol{\pi}^\mu$. One finds

$$^t\boldsymbol{\pi}^\mu = \frac{\partial L}{\partial \partial_0 \boldsymbol{\omega}_\mu} = E\left(\mathbf{E}^\mu \tilde{\mathbf{E}}^0 - \mathbf{E}^0 \tilde{\mathbf{E}}^\mu\right) \tag{107}$$

$^t\boldsymbol{\sigma}$ stands for the transpose of a 2×2 matrix $\boldsymbol{\sigma}$ again.

One deduces from this that $\boldsymbol{\pi}^0 = 0$. In addition one finds Ashtekar's celebrated result that vector potential and the "densitized" inverse dreibein are conjugate variables

$$\boldsymbol{\pi}^i = 2\tilde{\mathrm{e}}^i . \tag{108}$$

Proceeding further according to the rules in the book, the canonical Hamiltonian H_C is obtained as a sum of three terms, all of which are first class constraints. This means that they vanish on the constraint surface.

$$\begin{aligned} H_C &= -\underline{N} H - 2N^i H_i - 2G, && (109)\\ H &= tr\left(\mathbf{F}_{ij} \tilde{\mathrm{e}}^i \tilde{\mathrm{e}}^j\right), && (110)\\ H_j &= tr\left(\mathbf{F}_{ij} \tilde{\mathrm{e}}^i\right), && (111)\\ G &= tr\left(\boldsymbol{\omega}_0 D_i \tilde{\mathrm{e}}^i\right) \ . && (112) \end{aligned}$$

Summation are from 1 to 3. The hermitian conjugate $h.c.$ involves the adjoint matrices $\mathbf{F}^*_{\mu\nu}$ and $\boldsymbol{\omega}^*_0$; the dreibein matrices $\tilde{\mathrm{e}}^i$ are self adjoint.

This is the second one of Ashtekar's celebrated results. It exhibits the constraints in polynomial form. $\underline{N}$, N^i and the matrix $\boldsymbol{\omega}_0$ are Lagrange multipliers; they have zero conjugate variables. The covariant derivative has the conventional form

$$D_i\boldsymbol{\sigma} = \partial_i\boldsymbol{\sigma} + [\boldsymbol{\omega}_i, \boldsymbol{\sigma}]. \tag{113}$$

Time development is from space-like hyper-surface to space-like hyper-surface. There is freedom in choosing the foliation into hyper-surfaces and of Lorentz gauge transformations. This is reflected in the freedom of choice of the Lagrange multipliers. We choose

$$N^i = 0, \quad \boldsymbol{\omega}_0 = 0. \tag{114}$$

The field equations are obtained with the help of the following lemma. Suppose $\mathbf{A}^{ij}$ is independent of $\boldsymbol{\omega}_k$. Then

$$\frac{\delta}{\delta\boldsymbol{\omega}_k} tr\left(\mathbf{A}^{ij}\mathbf{F}_{ij}\right) = {}^t\left(D_j\mathbf{A}^{kj} - D_j\mathbf{A}^{jk}\right). \tag{115}$$

One obtains

$$2\dot{\boldsymbol{\omega}}_k = \frac{\partial H_C}{\partial\tilde{\mathrm{e}}^k} = \underline{N}[\mathbf{F}_{ki}, \tilde{\mathrm{e}}^i], \tag{116}$$

$$2\dot{\tilde{\mathrm{e}}}^k = -\frac{\partial H_C}{\partial\boldsymbol{\omega}_k} = \underline{N}D_j[\tilde{\mathrm{e}}^k, \tilde{\mathrm{e}}^j] \tag{117}$$

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