Universal Dynamics, a Unified Theory of Complex Systems. Emergence, Life and Death*

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Dedicated to the memory of Harry Lehmann

April 20, 2000

abstract A universal framework is proposed, where all laws are regularities of relations between things or agents. Parts of the world at one or all times are modelled as networks called SYSTEMs with a minimum of axiomatic properties. A notion of locality is introduced by declaring some relations direct (or links). Dynamics is composed of basic constituents called *mechanisms*. They are conditional actions of basic local structural transformations ("enzymes"): indirect relations become direct (friend of friend becomes friend), links are removed, objects copied. This defines a kind of *universal chemistry*. I show how to model basic life processes in a self contained fashion as a kind of enzymatic computation. The framework also accomodates the gauge theories of fundamental physics. Emergence creates new functionality by cooperation - nonlocal phenomena arise out of local interactions. I explain how this can be understood in a reductionist way by multiscale analysis (e.g. renormalization group).

1 Introduction

When we speak about the world, we speak about models of parts of the world which are constructed by the human mind. I postulate that they reflect the structure of human thinking as formulated in the following
preaxiom: *The human mind thinks about relations between things or agents*

*To be published in Commun. Math. Phys.

Relations will be interpreted as directed binary relations from a source to a target. Their constitutive property is that they can be composed - think of friend of a friend, brother in law, next nearest neighbor etc.

Traditionally, emphasis in physics has been on objects, like atoms or elementary particles. But relations are equally important. They integrate the objects into a network. In adaptive systems, the relations change in time in such a way that the connectivity of the whole network may change. Some mistaken views concerning reductionism or emergence result from neglect of the basic role of relations.

One may regard geometry as ancestor of relational theories. It knows a relation of parallelism between pairs of tangent vectors which serves to define straight lines and distances.

The modern theories of fundamental physics are relational theories. This is true for the established theories, general relativity and the standard model of elementary particle physics. They are geometric theories (gauge theories). It is also true of string theory [14] and of the loop space approach to quantum gravity in Ashtekar variables [2]. Strings, whether open or closed, can be composed when they touch appropriately, and similarly for loops.

A general plan of *relational biology* has been put forward by Rashevsky [61] and Rosen [63] decades ago.

There is a mathematical theory of relations, category theory. Lawvere sought a purely categorical foundation of all mathematics, including set theory [42]. The mathematical biologists used category theory from the start.

However, category theory lacks an essential ingredient of physical theories, *locality*. The fundamental physical theories are local in space time in the sense that the basic equations only relate quantities at (infinitesimally) close points in space, and at (infinitesimally) close instances of time. This is the celebrated Nahewirkungsprinzip which was discovered in the last century. Newton's theory does not obey it, but Einsteins general relativity, which supercedes it, does, and so does electrodynamics. The processing of chemically bound atoms and molecules in the living cell is mostly performed by biochemical enzymes, and their action is local - they act somewhere at a time.

Here I postulate a more general locality principle which does not refer to space. Certain relations are singled out as *direct relations*, called *links*, and all others are obtained from them by composition. The generalization is desirable for several reasons - systems not in space, investigations of properties of space time itself, quantum objects in quantum systems [1] whose parts are far apart in space, rapid communication over long distances (like in the Newtonian limit of general relativity) etc.

The fathers of artificial intelligence did not adopt locality as a default option. This lead to such problems as mentioned by Marvin Minsky [58] when he says that a robot needs to be told a lot of facts about its surroundings, for instance that the wall does not fall down when he paints the table in the middle of the room. Without a locality principle, complexity becomes unmanageable [59].

What is assumed is not explained. Therefore, a fundamental physical theory is the more fundamental the less a priori structure is assumed. And a theory of complex systems is the more general the less a priori structure is assumed. Here I propose a general framework which provides a minimum of a priori structure through the definition of a SYSTEM . It is in the spririt of L.v.Bertalanffy [7], the pioneer of general systems theory, and of Wittgenstein's *tractatus* [70]. Basically it defines in a precise way a notion of structure. Its axioms contain essentially no more than my preaxiom and locality. In contrast with automata theory [71], the framework is supposed to be self contained. In principle, there are no data in SYSTEMs other than structure, no states of any part of a SYSTEM other than structure and no information exists other than specification of structure. The miracle is how much can be modelled with so little building material. I show models of life processes which exemplify this. The selfcontainedness of the present framework makes it a natural candidate for implementation on a computer. Software has been written and will be presented elsewhere [55]. It offers the convenience that one may compose, record and run a program by mouse click, build models of complex systems in this way and simulate dynamical processes.

With locality, models become much more similar to those which physicists are used to. This results in a promising strategy to bring methods of theoretical physics to bear on very general complex systems, including biological and social systems.

Actually I want to model not only material parts of the world, but also space time and immaterial constructs of the human mind like proposition logic. Different kinds of SYSTEMs are distinguished by structural features which generalize what is known in physics as *constraints* on initial states. For instance, Gauss' law and gauge group isomorphic to $\mathbf{R}$ are constitu-

[1] They are not considered in this paper

tive constraints for an electromagnetic field. Space imbedded in space time is characterized by constraints which may be summarized by saying that there is geometry. And the constitutive property of matter is that it is in space time. Conservation laws need no be imposed in addition. They follow from requirements of internal consistency. Einstein's equations for space time would be inconsistent without covariant conservation of the energy momentum tensor, and Maxwell's equations of electrodynamics would be inconsistent without conservation of electric charge. For reasons of space, I cannot expand on these aspects here, cp. ref.[50]. But note that Darwinian evolution has competition for scarce resources as a constitutive feature, and scarcity is a consequence of conservation of matter and energy.

There are no numbers in the framework to begin with, but a numerical description can sometimes be obtained by coordinatization. For instance, a gauge group is always defined as a structural property of a SYSTEM (at one time) and through its coordinatization one obtains a numerical decription of the gauge fields in lattice gauge theory models of elementary particle physics. In fact, the constitutive feature of pure gauge theories - differential geometry on principal or associated fibre bundles - can be recovered from the axioms plus one single extra structural assumption, $forth \circ back = identity$, cp. ref.[49] and section 4.

Let us turn to dynamics. I consider local Markovian dynamics in discrete time. The dynamics is composed from special local structural transformations. They are atomic constituents of dynamics and will be called *enzymes*. They are the mathematical models not only of the aforementioned biochemical enzymes , but of any kind of agent which causes change locally. Besides, there are predicates which enquire about local structural properties and serve to formulate conditions to which the enzymes action is subject. The conditional action of an enzyme - i.e. a pair (enzyme, predicate) - is called a *mechanism*. Mechanisms are valuable tools in theoretical immunology [57].

It is not a trivial task to construct a well defined deterministic dynamics from mechanisms because their actions here and at a neighbouring location may not commute. Since Petri [60] this is known to computer scientists working on parallel computing as the *concurrency problem*. I solve it with the help of a generalization of the well known device of Jacobi sweeps [68]. The resulting theory may be viewed as a *universal chemistry* in which general objects and links substitute for atoms and their chemical bonds. [2] and

[2]Actually there is in chemistry another basic relation besides chemical bonds, spatial

dynamics can be interpreted as *enzymatic computation* [55]. The λ-calculus [31] can be implemented, therefore enzymatic computation can do anything a Turing machine can [62].

Enzymes share with matter the property of being somewhere. In biochemistry they are tied to material bodies, while in fundamental physics they are imagined to be ubiquitous [3]. We regard the presence of mechanisms as part of the specification of the initial state of a SYSTEM. The universal dynamics says: All mechanisms operate.

A SYSTEM *sub specie aeternitatis* - i.e its whole history - is again a SYSTEM. It is called a *drama.* The dynamics manifests itself as structural properties of this system.

The dynamics can be stochastic. In this case the drama is a random SYSTEM, and its links etc. are random variables. When the dynamics is sufficiently stochastic, the drama becomes a classical equilibrium statistical mechanical system, albeit "in one more dimension", with possible initial conditions now figuring as boundary conditions. There may also be external fields which represent interactions with an environment. Some emergent phenomena in biological systems can be modelled that way and they then appear as instances of familiar phenomena in equilibrium statistical mechanics such as restoration of spontaneously broken symmetries. Elsewhere [48] I illustrate this on the example of a very much simplified model of *schools of fish swimming in coherent array which abruptly turn together with no leader guiding the group.*

Emergence is generally understood as leading to new and often unexpected properties of a whole which are not shared by its isolated parts. [4] We regard a system as genuinely complex, if its properties are not all shared by subsystems with few objects. Emergent phenomena which arise in this way are nonlocal phenomena. Yet we want to understand them as a consequence

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proximity

[3] In the canonical approach to classical physics, the dynamics it determined by the Hamiltonian H according to $\xi \mapsto \xi + \{H, \xi\}\delta t$. In a field theory, the Hamiltonian is a sum of local pieces which act locally. Call them enzymes. They are composed from basic microenzymes - canonical variables q and p. We may imagine that these enzymes are everywhere, and they stay there. One might want to think of them as dynamical, capable of changing their location or composition (functional form) as a consequence of the dynamics. But that is impossible because whatever H may be, the Poisson bracket $\{H, H\} = 0$. In dissipative systems the situation is different.

[4] An illuminating discussion of the relevance of concepts in complexity, including emergence, for immunology can be found in I. Cohen's book [11]

of local interactions. Life is an emergent phenomenon. It involves emergent functionality. According to Maturana and Varela [56], living organisms are autopoietic - characterized by being able to make their own elements. [5] Typically this involves creation of structure by copying or translation from templates, and preparation of building blocks by digestion, i.e. degradation of structure. The action of the splitFork-enzyme of section 3.1.1 is an example of emergent functionality. The whole enzyme can copy arbitrary SYSTEMs, but the individual microenzymes from which it can be composed cannot. And when one of them is missing or carries the wrong predicate, the copy-functionality is also lost.

Some emergent phenomena like wave propagation can be understood by exploitation of symmetry and linearity or other special methods. Otherwise, a multiscale analysis is called for. Although a genuinely complex system $\mathbf{S}$ cannot be understood as a whole by looking at it locally, a complexity reduction is often possible by local considerations. This was the central idea of Wilsons renormalization group [69, 38]. One constructs an effective theory, i.e. a description in terms of a new system $\mathbf{S}^1$ with new objects which represent subsystems, but retain only as much information on their internal structure as is relevant for their cooperation. Links between them are also constructed. Enzymes may be attached which represent functionality of compound enzymes at the smaller scale. The resulting system is still complex, but may have much fewer degrees of freedom. Then the procedure of complexity reduction may be iterated, leading to a multilevel description.

Mathematically, the chief insight is the relation between block spin constructions in a renormalzation group (RG) setup, and collections of dual colimits and factorizing cones in categories. Experience with the rigorous renormalization group approach to gauge theories [5] is valuable because general SYSTEMs share many features of lattice gauge theories, cp. section 4. In what way they are essentially more general is best seen in the section 4.1 on logic. Monte Carlo RG-studies of gauge theories have also been performed, cp. e.g. [29], and the Monte Carlo RG-method is still being improved [9].

[5] Maturana and Varela thought of biological systems only, but Luhmann [46] generalized the notion of *Autopoiesis* to social systems.

2 Structure

This section and the next introduce the basic mathematical framework. Parts of the world at one time as well as their whole histories are modelled as SYSTEM's with certain axiomatic properties. A more descriptive name would be "local category".

According to L. van Bertalanffy, [7], *a system is a set of units with relationships between them.* I precisize.

Definition 1 (SYSTEM) *A* SYSTEM *is a model of a part of the world as a network of* objects $X, Y, ...$ *(which represent things or agents) with* arrows $f, g, ...$ *which represent directed relations between them.*
One writes $f : X \mapsto Y$ *for a relation from a* source *(domain)* X *to a* target *(codomain)* Y.

The arrows are characterized by axiomatic properties as follows:

1. composition. *Arrows can be composed. If* $f : X \mapsto Y$ *and* $g : Y \mapsto Z$ *are arrows, then the arrow*

$$g \circ f : X \mapsto Z$$

   is defined. The composition is associative, i.e. $(h \circ g) \circ f = h \circ (g \circ f)$.

2. adjoint. *To every arrow* $f : X \mapsto Y$ *there is a unique arrow* $f^* : Y \mapsto X$ *in the opposite direction, called the adjoint of* f. $f^{**} = f$ *and* $(g \circ f)^* = f^* \circ g^*$.

3. identity. *To every object* X *there is a unique arrow* $\mathbf{1}_X : X \mapsto X$ *which represents the identity of a thing or agent with itself.*

$$\mathbf{1}_X = \mathbf{1}_X^*, \qquad \text{and} \qquad \mathbf{1}_Y \circ f = f = f \circ \mathbf{1}_X$$

   for every arrow $f : X \mapsto Y$.

4. locality*: Some of the arrows are declared* direct *(or fundamental); they are called* links. *All arrows* f *can be made from links by composition and adjunction,* $f = b_n \circ ... \circ b_1, (n \geq 0)$ *where* b_i *are links or adjoints of links; the empty product* $(n = 0)$ *represents the identity.*

5. composites: *The objects* X *are either* atomic *or* SYSTEMs. *In the latter case,* X *is said to have internal structure, and the objects of the* SYSTEM X *are called its* constituents.

6. non-selfinclusion: *A* SYSTEM *cannot be its own object or constituent of an object etc. Ultimately, constituents of ... of constituents are atomic.*

The links and objects of a system are called its elements.
A system is called connected *if there are arrows to all other objects from some (and therefore all) objects.*
A system is called unfrustrated *if there is at most one arrow from X to Y for any objects X, Y.*

Axioms 1 and 3 are those of a category. Ignoring specification of links and the $*$-operation, a SYSTEM $\mathbf{S}$ becomes a category $Cat(\mathbf{S})$.

There is a long standing controvercy in philosophy concerning identity, see e.g. Wittgenstein [70], Satz 5.303 or Quine [72]. In SYSTEMs theory, the identity arrows $\mathbf{1}_X$ are as important as the number 0 in arithmetics. Later we shall have occasion to introduce also special arrows between two objects which are identical in the sense of indistinguishable (i.e. copies). By abuse of language they will also be called *identity arrows*.

The idea of an adjoint (axiom 2) is that a relation in the opposite direction should be specified in some way by any link. There can be different ways in which links can be adjoint. For instance, if objects X, Y are SYSTEMs, hence categories, and $f^* : Y \mapsto X$ is left adjoint functor of a functor f (s. later) then $f^{**} = f$ is right adjoint functor of f^*.

Axiom 4 introduces locality as explained in the introduction.

Axiom 5 makes the whole scheme self contained. And according to Jacob [35] *Tout objet que considère la biologie represente un système de systèmes.*

The totality of statements about SYSTEMs which are meaningful as a consequence of the axioms will be called the *language of thought*.

Assumption 1 *Unless otherwise stated, it is assumed that constituents, constituents of constituents etc. of objects of* $\mathbf{S}$ *are not objects of* $\mathbf{S}$.

It must be emphasized that atomicity of an object is not a property of something in the world, but of a particular model which describes some of its aspects on some scale.

Objects with internal structure are *black boxes*. Later on we shall "dissolve" such objects, making their interior structure visible by putting links from some of their constituents, and thereafter they may be treated as atomic although they still stand for the "same" (composite) object.

Definition 2 (Types of links) *A link $b : X \mapsto Y$ is said to be invertible if there exists an arrow, denoted b^{-1} such that $b \circ b^{-1} = \mathbf{1}_X$, and $b^{-1} \circ b = \mathbf{1}_Y$.*

It is said to be unitary *if $b^* = b^{-1}$.*

Links whose adjoints are links will be called bidirectional *for short.*

Sequences $b_1, ..., b_n$ of links or adjoints of links which can be composed are said to make *a path of length $n \geq 0$*. Composability requires that the target of b_i is the source of b_{i+1}. The length of the shortest connecting path can be used to measure distance between objects.

Definition 3 (Subsystems) *A* subsystem $\mathbf{S}_1$ *of a* SYSTEM $\mathbf{S}$ *is generated by a set of objects in* $\mathbf{S}$ *and a set of links in* $\mathbf{S}$ *between these objects. Its arrows are all arrows in* $\mathbf{S}$ *that can be composed from these links and their adjoints.*

The boundary *of* $\mathbf{S}_1$ *consists of the links in* $\mathbf{S}$ *with target in* $\mathbf{S}_1$ *which are not links or adjoints of links in* $\mathbf{S}_1$*.*

The environment *of* $\mathbf{S}_1$ *is the system generated by the objects of* $\mathbf{S}$ *not in* $\mathbf{S}_1$ *and the links between them.*

For $n \geq 1$, the n-neighborhood of an object X is the subsystem which contains all objects connected to X by a path of length $\leq n$, and is generated by all links between them. Identity links in the path are counted as contributing 0 to its length.

A SYSTEM *is called* locally unfrustrated *if all its* 1*-neighborhoods are unfrustrated subsystems.*

Note that it is *not* required that adjoints of links in $\mathbf{S}_1$ which are links in $\mathbf{S}$ are also links in $\mathbf{S}_1$.

Following Luhmann [46], one may also want to consider the *internal environment* $\mathbf{I}_1$ of $\mathbf{S}_1$. It is the system whose objects are the constituents of non-atomic objects X in $\mathbf{S}_1$ and whose links are the links between them. Arrows are equivalence classes of paths, equivalences being determined by the equivalences in the SYSTEMs X. Given our standing assumption 1, $\mathbf{I}_1$ is *not* a subsystem of $\mathbf{S}$.

The brick wall shown in figure 1 is an example of a locally unfrustrated system, and so is any triangulated 2-manifold, with the 1-simplices as links.

2.1 Structure preserving maps

As always in physics, we shall not distinguish between isomorphic systems. To make this precise we need to consider structure preserving maps called

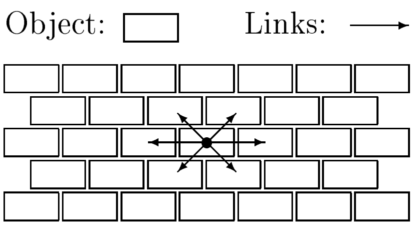

Figure 1: Brick wall. The links b are translations of a brick to a nearest neighbour's position. The arrows are equivalence classes of paths subject to the equivalences $b^* = b^{-1}$ and $b_1 \circ b_2 \circ b_3 = \mathbf{1}_X$ for any triangular path from X to X. The system is locally unfrustrated, but would become (globally) frustrated if the wall were closed to a round tower.

local functors. [6]

In category theory, a *functor* $F : \mathbf{C} \mapsto \mathbf{C}'$ is a map of the objects of a category $\mathbf{C}$ to objects of a category $\mathbf{C}'$ and of arrows f of $\mathbf{C}$ to arrows of $\mathbf{C}'$ such that source and target of $F(f)$ are the images of source and target of f, and

$$\begin{aligned} F(f \circ g) &= F(f) \circ F(g), \quad (1) \\ F(\mathbf{1}_X) &= F_{\mathbf{1}_{F(X)}} \,. \quad (2) \end{aligned}$$

In contravariant functors, eq.(1) is replaced by

$$F(f \circ g) = F(g) \circ F(f), \quad (3)$$

Definition 4 (local functor) *A local functor $F : \mathbf{S} \mapsto \mathbf{S}'$ is a map of a* SYSTEM $\mathbf{S}$ *into another* SYSTEM $\mathbf{S}'$ *which obeys the above requirements on a functor of categories, maps links into links, and obeys*

$$F(f^*) = F(f)^* \quad (4)$$

In a contravariant local functor, eq.(3) is substituted for eq.(1).

An isomorphism *of* SYSTEM*s is a local functor whose inverse exists as a local functor. An* anti-isomorphism *of* SYSTEM*s is a contravariant local functor whose inverse exists as a contravariant local functor.*

[6] This makes precise what Wittgenstein leaves undefined in his isomorphism theory in *tractatus* 2.15 when he postulates that the images of two objects are related "in the same way" as the objects.

No local functor between two SYSTEMs need exist unless the identity arrows of the image are links. (Typically they are not.)

Anti-isomorphisms relate complementary shapes. They are important in cognition. The surface (boundary) of a lock and a key have opposite orientation; this will be reflected in an anti-isomorphism. This is important for life. In biochemistry, the specificity of enzymes for particular substrates is due to a lock-key match of parts of their surfaces. And receptors on cell walls function according to the same lock-key principle [1].

It is important to note that the internal structure of black boxes (nonatomic objects) is declared irrelevant by not distinguishing isomorphic SYSTEMs, and so is the distinction between atomic and nonatomic objects. The only usage of the internal structure is in constructing links of the SYSTEM and their composition $\circ$. One does not look into black boxes anymore once they are in place.

If isomorphism of corresponding nonatomic objects is demanded, we speak of a *strong isomorphism.*

Categorical Remark 1 *Local functors define the category of* SYSTEM*s whose objects are* SYSTEM*s and whose arrows are local functors.*

Among the isomorphisms $F : \mathbf{S} \mapsto \mathbf{S}'$, there is a particularly important class called *gauge transformations.*

They are "inner" in the sense that they are generated by arrows of the system. Since gauge transformatons are isomorphisms, statements about a SYSTEM in the language of thought are necessarily gauge invariant. It follows that observables must be gauge invariant.

More precisely, one may define objective observables to be boolean functions on SYSTEMs, well defined for all SYSTEMs, while subjective observables require for their definition specification of a distinguished object X (the subject), and possibly some links b with target X (e.g. the direction in which a speaker points). Objective observables must be gauge invariant, while for subjective observables this needs only be true for gauge transformations which are trivial at X and don't change its specified links b.

Given $\mathbf{S}$ and an object X of $\mathbf{S}$, the group G_X of local gauge transformations consists of all invertible arrows $g : X \mapsto X$. (It is also called the holonomy group). Its identity element is $\mathbf{1}_X$.

Gauge transformations take $\mathbf{S}$ into a system $\mathbf{S}'$ with $Cat(\mathbf{S})= Cat(\mathbf{S}')$.

Definition 5 (gauge transformations) *A gauge transformation F is defined by selecting an element $g_X \in G_X$ for every object X, and mapping arrows $f : X \mapsto Y$ into*

$$F(f) = g_Y^{-1} \circ f \circ g_X. \tag{5}$$

*The links of $\mathbf{S}'$ are the images of links of $\mathbf{S}$, and the *-operation $^+$ in $\mathbf{S}'$ is defined in terms of the *-operation in $\mathbf{S}$ by*

$$f^+ = \sigma_X^{-1} \circ f^* \circ \sigma_Y \tag{6}$$

with $\sigma_X = g_X^* \circ g_X$. A gauge transformation is called unitary *if $g_X^* = g_X^{-1}$ for all X.*

One verifies that the conditions for an isomorphism are satisfied. The composition law is preserved. The new star operation satifies $f^{++} = f$ and $\mathbf{1}_X^+ = \mathbf{1}_X$ as it should, and $F(f^*) = F(f)^+$.

Theorem 1 (Gauge group) *In connected systems $\mathbf{S}$ where all links are unitary, all gauge transformations are unitary, and all groups G_X of local gauge transformations are isomorphic. Their isomorphism class G is called the* gauge group.

Proof: With all links, all arrows are unitary. Therefore all $g_X : X \mapsto X$ are unitary. Given X, Y, there exists an arrow $f : X \mapsto Y$ because $\mathbf{S}$ is connected. If $g_X \in G_X$ then $g_Y = f \circ g_X \circ f^* \in G_Y$, and conversely. Therefore G_X and G_Y are isomorphic. q.e.d.

Gauge transformations in electrodynamics are the standard example. Electrodynamics will be considered in section 3.2 later on. Here is another example

Example 1 (Fundamental group) *Consider the system made from a triangulated manifold as follows. The objects are the 0-simplices and the links are the 1-simplices; adjunction is change of orientation, adjoints of links are links. The arrows are equivalence classes of paths $b_0, ..., b_n$, defined by the following equivalence relation:*

- all links are unitary,

- $b_0 \circ b_1 \circ b_2 = \mathbf{1}_X$ if b_0, b_1, b_2 is a closed path from X to X (a triangle).

The gauge group is the fundamental group of the manifold.

Theorem 2 (Reconstruction of objects) *A* SYSTEM *is determined up to isomorphism if*

- the arrows are enumerated or given as a set,

- it is specified which arrows can be composed and which arrow is the result

- the links among the arrows are specified.

- the adjoints of arrows are specified.

Conversely, any such data determine a SYSTEM *if all the arrows can be obtained by composing links and their adjoints, and if the *-operation satifies the consistecy conditions imposed by the axioms.*

PROOF The corresponding result in category theory is standard [40]. One reconstructs the objects as equivalence classes X of links, two links b_1 and b_2 being equivalent if there exists an arrow f such that $b_1 \circ f$ and $b_2 \circ f$ are both defined, X is then the common target of all links equivalent with b_1. f with target X is equal to the identity $\mathbf{1}_X$ if $g \circ f = g$ whenever it is defined.

Categorical Remark 2 *Gauge transformations are invertible functors which preserve objects and admit a natural transformation to the identity.*

2.2 Tautological character of the axioms

I wish to convince the reader that there is no more in the axioms than what is intended by the preaxiom, and locality. The composition law and adjunction need to be discussed.

The axiomatic properties assure that a SYSTEM is a directed (pseudo-multi) graph if the number of objects is finite. The edges of the graph are the links and its vertices are the objects. The graph may have edges which are loops (pseudographs), and it can have multiple edges between the same vertices (multigraph) [28]. Deviating from standard nomenclature, I will also speak of a graph when the number of vertices is countable.

Let us first turn to adjoints. One may consider the existence of the fundamental relation $b : X \mapsto Y$ as a directed relation from Y to X. This amounts to introducing formal adjoint links in the graph in the opposite direction.

Given a graph Γ, which represents basic relations between things or agents, we have no composition rule. But one can make from Γ a category $\mathbf{S}_\Gamma$ in a universal way. This yields a SYSTEM. The arrows from X to Y

are the paths from X to Y made from links and their formal adjoints. All the SYSTEMs with given directed graph Γ are obtained from $\mathbf{S}_\Gamma$ by passing to equivalence classes of paths. So, *the freedom of choosing the composition law merely introduces the option of waiving distinctions between relations.*

Similarly, there is a universal way of making a category $\mathbf{U}_\Gamma$ with unitary links from any given directed graph. It is obtained from $\mathbf{S}_\Gamma$ by imposing the relations $b \circ b^* = \mathbf{1}$ and $b^* \circ b = \mathbf{1}$ i.e. considering only nonbacktracking paths. Every SYSTEM with unitary links is obtained from $\mathbf{U}_\Gamma$ by passing to equivalence classes of nonbacktracking paths.

Categorical Remark 3 *The passage to equivalence classes of arrows defines a unique local functor. Therefore we have*

Theorem 3 *The forgetful functor F from the category of finite* SYSTEM*s* $\mathbf{S}$ [*resp. finite* SYSTEM*s with unitary links*] *to the category of directed (pseudomulti-)graphs* Γ *has a left adjoint functor* $F^* : \Gamma \mapsto \mathbf{S}_\Gamma$ [*resp.*$\Gamma \mapsto \mathbf{U}_\Gamma$].

3 Universal dynamics

Dynamics shall be compose from local structural transformations. They are special graph transformations [64] obeying some strict locality requirements. They are universal in the sense that their action is defined for arbitrary systems the same will therefore be true for dynamics composed from them[7].

We shall distinguish four kinds of such transformations

motion

growth

death

cognition.

They are reversible except for death. I discuss them one by one.

Motion promotes indirect relations to direct relations. Either arrows composed of two (or more) links are promoted to the status of a link (e.g. friend of a friend becomes friend), or a non fundamental adjoint becomes a link. The opposite, demotion of an adjoint link to non-link is also subsumed under motion.

[7]Univeral dynamics is intended to be universal also in the sense of universal constructions in category theory. But the appropriate theorems have not been proven yet.

Equations of motion in physics like the Maxwell equations (s. below) determine motion in this sense. Catalysis of bonds in chemistry (and elsewhere, figure 2) is motion in this sense (supplemented with removal of some link(s)), and so is motion in space. Let material body A be "at" space point C and C be "neighbour_of" B. If the relation "at$'$ =neighbour_of $\circ$ at" becomes fundamental instead of "at", it means that body A has moved from C to B.

The category $Cat(\mathbf{\Sigma}_t)$ does not change at all in this kind of time evolution. Therefore we have

Theorem 4 *Any quantity* Q *which is determined by the category* $Cat(\mathbf{\Sigma}_t)$ *is a constant of* motion.

Growth copies objects.

Death removes links (together with their adjoints), or removes objects together with all links incident on them.

Cognition creates links between objects with matching internal structure. The match is supposed to be established by enzymatic computation, ultimately with enzymes of the other kinds. Identity links (cp. after definition 1) between atomic copies of objects are also admitted and serve as prototypical examples of cognitive links. The creation of links by cognition (cognitive links for short) is fundamentally different from creation of links from existing links by composition or adjunction because it creates *new* arrows. This will be important later on when we adapt Baas' distinction between deductive and observational emergence [3]. In principle, operations of this kind would also be mathematically well defined if they are nonlocal, but we don't want to admit that. If only objects within one 1-neighborhood can be linked, they must be connected before by a path of length at most 2. But the new arrow is not the arrow which this path defines. For instance, in chemistry spatial

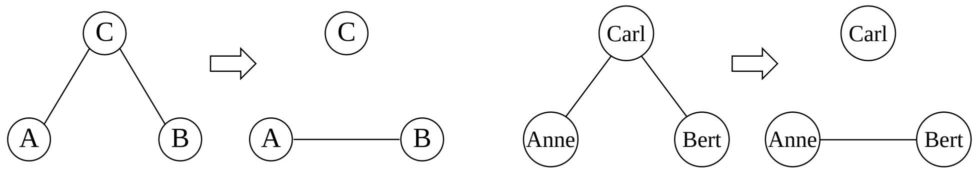

Figure 2: Catalysis in chemistry and elsewhere. A catalyst `C` binds molecules `A` and `B`. First a substrate-enzyme complex is built, where `A` and `B` bind to `C`. Next the composite arrow from `A` to `B` becomes fundamental.

proximity links are a prerequisite for forming chemical bonds. Receptors on membranes of living cells operate in a cognitive way, using lock-key type matching.

Before proceeding to formal developments, I will discuss some examples.

Copy processes are most important for life. Autopoietic systems make their own elements. Ultimately they make them from constituents or constituents of constituents etc. which are conserved material entities, not created. So the "making" is supply of structure, typically from templates. DNA is copied, it is also copied into RNA. RNA is copied and is also translated into sequences of amino acids, i.e. proteins.

The material constituents must also be supplied. Plants make organic material from anorganic substances and light by enzymatic action, but animals need to get their building blocks from organic materials breaking down its preexisting structure. Call this digestion.

I will demonstrate how copy processes and digestion can be modelled using no more than what is provided by the axioms, definition 1, and also how relaxation sweeps through extended SYSTEM's can be modelled.

3.1 Examples

3.1.1 Copying. The asymmetrical replication fork

There is an example of a local dynamics that can be used to produce within a finite time two copies of any finite SYSTEM whose links are all bidirectional. It is a mathematical abstraction and generalization of the asymmetrical replication fork mechanism which copies DNA in the living cell, see standard text books [1, 43]. During the copy process, links without fundamental adjoints appear.

A *fork* at X shall be a pair of links without fundamental adjoints with source and target X, respectively. The splitfork-action s_X shall be a local structural transformation. Figure 3.1.1 shows its action on chains (of pairs of directed links) like the DNA double helix. The generalization is as follows. A link is called "bidirectional" if its adjoint is a link, and "unidirectional" otherwise.

1. A copy X' of X is made.

2. The links incident on X other than loops are distributed among X and its copy as follows:

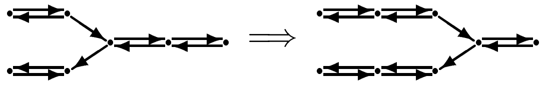

Figure 3: Action of the splitfork-enzyme s_X at X, for chains like DNA. The same mechanisms can operate on general SYSTEMs

- bidirectional links with target X get X' as their target
- unidirectional links with target X retain X as their target
- bidirectional links with source X retain X as their source
- unidirectional links with source X get X' as their source

The loops $X \mapsto X$ remain in place and get a copy $X' \mapsto X'$.

3. The adjoints of formerly unidirectional links are promoted to the status of links.

Theorem 5 (Universal copy constructor) *Let* $\mathbf{S}_0$ *be obtained from a finite connected system* $\mathbf{S}$ *whose links are all bidirectional by action of* s_{X_0} *at some* $X_0 \in \mathbf{S}$. *For* $t > 0$, *let* $\mathbf{S}_t$ *be obtained from* $\mathbf{S}_{t-1}$ *by action of* s_X *for all objects* X *which have forks.*

$\mathbf{S}_t$ *is well defined for* $t \geq 0$. *For sufficiently large* t, *it is independent of* t *and consists of two disconnected* SYSTEM*s, both isomorphic to* $\mathbf{S}$.

"Once replication has started, it continues until the entire SYSTEM has been duplicated". Upon substituting "genome" for "SYSTEM" this becomes a quote from a genetics text book [43].

Remark 1 *Copying may be initiated at several sites* X_0,... X_n *which are not connected by links.*

The action of the splitFork-enzyme is quite robust against errors due to computer failures which mimick local mutations. But a third copy is made of part of the SYSTEM *when a fundamental adjoint gets lost (or added) "at the wrong moment"* [8].

[8] In man, errors in copying the genome may result in Down's syndrome, the presence of three copies of chromosome no. 21 instead of the usual two

The theorem was first demonstrated in [51]. The fact that s_X is well defined requires a comment - what does it means that "a copy X' of object X is made"? Theorem 2 can be invoked to describe $s_X\mathbf{S}$ up to isomorphism. The isomorphism class does not retain the information about the internal structure of nonatomic objects. But this information can be retained by the copies, if desired. To do so, one uses the universal copy constructor to copy the objects which are themselves SYSTEMs, to copy their nonatomic constituents, and so on. Because of the axiom of non-selfinclusion this does not lead to an infinite recursion. In conclusion, if s_X is defined in this way, the stronger version of theorem 5 holds true where the phrase "both isomorphic to $\mathbf{S}$" is replaced by "both strongly isomorphic to $\mathbf{S}$".

s_X is an example of a local structural transformation ("enzyme"). On SYSTEMs as occur as $\mathbf{S}_t$ in theorem 5, actions s_X, s_Y commute. But this is a lucky circumstance.

The production of a copy is an "emergent phenomenon", a nonlocal phenomenon which arises from local interactions. New functionality -copying- emerges.

Categorical Remark 4 *s_X can be decomposed into several "microenzymes" which act in sequence. If all links are unitary, each of them specifies a functor of categories, albeit the trivial one, but the first of these functors is not surjective because an extra object is created.*

Such situations are admitted in the work of Ehresmann and Vanbremeersch [19] to which we turn later. It is not true that the functor *is* the dynamics in their work.

The decomposition is shown in ref.[55]. The first microenzymes is an elementary copy process which creates a duplicate of X and links it by an identity link to the original. The other microenzymes compose links and remove the identity link again.

When the copied object X is an indestructible material constituent, its equal may be imagined to be recognized in the environment (by a cognitive process) and absorbed by linking it to X by an identity arrow. .

3.1.2 Digestion

Consider a finite connected system $\mathbf{S}_0$ with bidirectional links and with a distinguished object X. I describe a local structural transformation d_X which

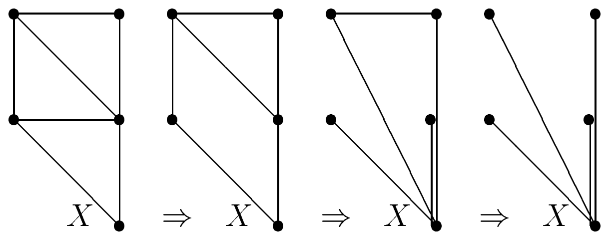

Figure 4: Digestion enzyme attacks at X

continues to act on a 1-neighborhood of X. Starting with an arbitrary $\mathbf{S}_0$, the action becomes trivial after some time and there results a system $\mathbf{S}$ with the same objects X, $Y_1, ..., Y_n$ as $\mathbf{S}$ but whose structure is completely degraded in the sense that its only links are one link from Y_i to X for each i, and their adjoints. (These links could be removed, but then "the food $\{Y_i\}$ is lost" since there are no relation to it anymore.)

d_X consists of consecutive steps.

1. (Death) The far side of all triangles of 3 links with tip X is removed, together with its adjoint.

2. (Motion) If b is a link from $Y \neq X$ to X and b' is a link from Y' to Y then $b \circ b'$ becomes a link and b' ceases to exist as a link.

3. Fundamental loops $X \mapsto X$ are removed.

Actually the 3rd step can be omitted when step 2 operates also for $Y = X$. Figure 3.1.2 illustrates the procedure.

Categorical Remark 5 *d_X specifies a functor of categories, albeit the trivial one, if the links are invertible, e.g. unitary.*

3.1.3 Sweeps through a SYSTEM

The splitFork copy procedure of section 3.1.1 relies on the propagation of a shock wave. The shock wave is the boundary between the part of the system which has already been copied, and the rest. It is made of links without fundamental adjoints. [9] The objects X at the outside of the boundary are

[9] Boundaries of subsystems which are distinguished in this or other ways serve as mathematical models of membranes in cell biology [1]

copied, and then the boundary passes past them. Instead of copying X, one may act on its neighbourhood with some (other) enzyme. In this way a sweep through the SYSTEM is generated which invokes an updating of the neighbourhoods of all objects in the SYSTEM.

3.2 Systems *sub specie aeternitatis*

There are two different ways of looking at dynamics. I will now examine the possibility of looking at a system *sub specie aeternitatis*, i.e. its whole history. This will also be viewed as a SYSTEM. Following Sorin Solomon's suggestion, I call it a *drama*.

Definition 6 (Drama) *A drama is a system* $\mathbf{S}$ *which is composed from subsystems* $\mathbf{S}_t$ *labelled by* $t = 0, \pm 1, \pm 2, ...$ *and links* e *in one or both directions between objects* $Y \in \mathbf{S}_{t+1}$ *and* $X \in \mathbf{S}_t$*. If there is such a link,* Y *is said to be descendent of* X*, and* X *is an ancestor of* Y*. It is required that every* $Y \in \mathbf{S}_{t+1}$ *is descendent of at least one object in* $\mathbf{S}_t$*. If there are several such objects, they must be connected by identity links.*

We will impose the additional condition that the time links are all unitary, and that any two paths between the same target X and source Y which are made exclusively of time links define the same arrow. It is a subtle question whether one might want to generalize this. If the condition is satisfied, the time links can be "gauged away", so that the parallel transporters along time links are trivial. This is familiar in gauge theory under the name "$A_0 = 0$ gauge" [32].

By definition, a drama is a SYSTEM. Its consideration *converts function into structure.* Dynamical laws constrain the structure of the drama.

Now we are ready to consider deterministic dynamics. Stochastic dynamics operates in the same way except that enzymes operate with certain probabilities.

An initial state $\mathbf{\Sigma}_t$ of an N-th order dynamics shall be a subsystem $\mathbf{S}_{[t,t+1-N]}$ of a drama which is generated by subsystems $\mathbf{S}_t, ..., \mathbf{S}_{t+1-N}$ and the time links beetween them, and enzymes that are attached to the objects and links of this SYSTEM. The enzymes code for the constraints on the local structure of the drama which determine $\mathbf{S}_{t+1}$ in terms of $\mathbf{S}_{[t,t+1-N]}$. The enzymes must

1. determine the descendents in $\mathbf{S}_{t+1}$, possible identity links between them and the existence of time links to and from their ancestors.

2. determine noncognitive links in $\mathbf{S}_{t+1}$. They are arrows in $\mathbf{S}_{[t,t+1-N]}$ composed with time links between $\mathbf{S}_t$ and $\mathbf{S}_{t+1}$

3. determine possible cognitive links between objects in $\mathbf{S}_{t+1}$ which are descendents of nonatomic objects in $\mathbf{S}_t$.

4. put enzymes on objects and links of $\mathbf{S}_t$.

Enzymes of *motion* make exactly one descendent of every object, with a pair of timelinks between it and its ancestor. They only make non-cognitive links.

A *growth* enzyme makes two or more descendents of one object or, in case of fusion, makes one common descendent of several objects which are linked by identity links.

death follows from absence of enzymes that make appropriate objects and links.

I pause to explain the notion of *cognitive links* which link two nonatomic objects. By definition, they are SYSTEMs $\mathbf{X}_1$, $\mathbf{X}_2$. A cognitive link is a local functor $f : \mathbf{X}_1 \mapsto \mathbf{X}_2$. Links between isomorphic SYSTEMs are the prototypical examples. If there is at most one link between two objects in $\mathbf{X}_1$, then the functor is determind by the images $f(Y)$ of objects Y in $\mathbf{X}_1$. We use special links -e.g. identity links - to connect Y and $f(Y)$. These identity links are supposed to be detemined by enzymatic computation, i.e. by a dynamical process in a SYSTEM which is generated by $\mathbf{X}_1, \mathbf{X}_2$ and an initial tentative identity link between some constituents of these. The making of cognitive links only makes sense if the dynamics determines future SYSTEMs up to *strong* isomorphism, because the internal structure of nonatomic objects matters.

For illustration of the relation between drama and dynamics, consider the splitFork-dynamics. The $t+1$ piece of Figure 6 shows a portion of a drama.

Another example are the discretized Maxwell equations on a cubic lattice and in discrete time. This dynamics is of second order, so the dynamical laws will involve links in 3 layers $\mathbf{S}_{t+1}, \mathbf{S}_t$ and $\mathbf{S}_{t-1}$ and time links. In addition there are constraints on the initial state, they involve links in $\mathbf{S}_t$ and $\mathbf{S}_{t-1}$ and time links between. All these constraints on the structure of the drama have the form

$$l = \mathbf{1}_X \tag{7}$$

where l are arrows $X \mapsto X$ that are made from closed paths with the above links. They are shown in figure 5.

We explain below why these are the Maxwell equations including the Gauss constraint.

Let us show that the dynamics is well defined. The links are unitary and the dynamical laws involve closed paths l with exactly one link b in $\mathbf{S}_{t+1}$. Therefore they have a unique solution

$$b = u^* \quad \text{if } l = b \circ u \tag{8}$$

where u is composed of the remaining links in the path p. The time links in path u are gauged away and the remaining ones are determined by the initial state.

We see that the time evolution merely amounts to promoting arrows u^* of the system to the status of link, while the old links may lose that status but remain as a arrows. In other words, the category $Cat(\mathbf{\Sigma}_t)$ does not change at all. Recall that this is always the case in motion.

The Gauss constraint is preserved in time. This will be shown in section 6 using tools of noncommutative differential calculus.

The gauge group $\mathbf{R}$ of electrodynamics is determined by the initial state; it is also counted as a constraint on the initial state. The coordinatization of links by real vector potentials (below) comes from the gauge group $\mathbf{R}$ similarly as in lattice gauge theory. This can be deduced from the representation theorems to be proven in section 4.

Therefore real values are attached to the links. If the lattice gauge field comes from a vector potential $\mathbf{A}$ in the continuum they are

$$\bullet \to \bullet \quad = \quad \int \mathbf{A} d\mathbf{x}, \qquad \text{whence} \tag{9}$$

$$\Box \quad = \quad \oint_{\Box} \mathbf{A} d\mathbf{x} = \int_{F:\partial F = \Box} \mathbf{B} d\mathbf{f} \tag{10}$$

for loops around squares. $\mathbf{B}$ is the magnetic field. Parallel squares are surrounded with opposite orientation. Therefore the total contribution of paths around all spacelike plaquettes going through $\to$ is $\approx \nabla \times \mathbf{B} \cdot$ area, while a timelike plaquette gives something proportional to the electric field, since $-\mathbf{E} = \dot{\mathbf{A}}$. Putting everything together we get Maxwell's equation $\dot{\mathbf{E}} = \nabla \times \mathbf{B}$, and Gauss' law $\nabla \mathbf{E} = 0$. The other Maxwell equation follows from the existence of a 4-vector potential.

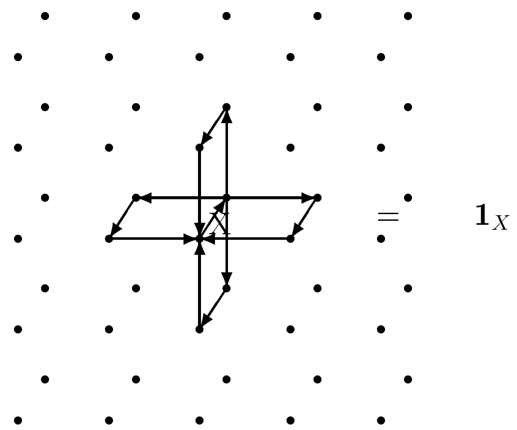

Figure 5: Maxwell Drama. For the Gauss-constraint, the direction to the back is the time direction. For the equations of motion, the upward direction is the time direction. The equations say that the parallel transporter along the path around all the displayed plaquettes equals the identity. Their is one equation for every pair of links. Formally, the Yang Mills equations are of the same form. Only the gauge group is different

Charged fields can be put in [50]. Note that charge conservation is required by the internal consistency of the Maxwell equations - indestructibility of charged matter is buildt into a structural description, it need not be postulated separately.

The Yang Mills equations of elementary particle physics have the same form, at least formally. Only the gauge group is different. Higgs physics can also be put in, at a prize [49]. The world is regarded as two sheeted, one sheet carries the left handed matter and the other the right handed matter. The two 4-dimensional sheets might be boundaries of a five (or higher)-dimensional world. The Higgs fields are possibly nonunitary parallel transporters between the sheets, as in the model of Connes and Lott [13], but with conventional locality requirements, cp. section 6. The prize is that each sheet should have its own gauge group G_L resp. G_R. The strong gauge group is therefore $SU(3) \times SU(3)$, broken spontaneously by a Higgs to $SU(3)$. But this breaking is expectedto produce a massive vector meson, the axigluon [24] which has not been found until now.

3.3 Concurrency

This section presents details on how a well defined dynamics can be composed from enzymes. There is a technical problem which is known to computer sci-

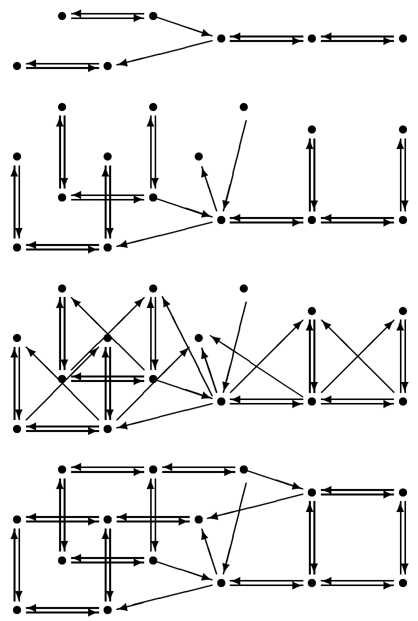

Figure 6: splitfork dynamics, concurrent version. The time $t+\frac{3}{4}$ step is not shown

entists as the concurrency problem. Enzymes specify local structural transformations, but the action of such transformations at neighboring locations may not commute.

The drama point of view amounts to solving the concurrency problem by a generalization of what is known in applied mathematics as a Jacobi sweep (as opposed to Gauss Seidel sweeps)[68]. In a Jacobi sweep one updates variables attached to nodes and links of a grid by visiting them one at a time and determining the values of their particular variables at time $t+1$ in terms of time t values of variables attached to objects and links in the neighborhood. The result is independent of the order of visits.

For simplicity consider first order dynamics, and ignore the possibility of making cognitive links. Given $\mathbf{S}_t$, the objects of $\mathbf{S}_{t+1}$ need to be made as descendents, and the links in $\mathbf{S}_{t+1}$ need to be made.

Divide the time interval in four and let the production of descendents take place at time $t+\frac{1}{4}$ and the initial production of new links at time $t+\frac{1}{2}$. Suppose that the production of descendents and their time links is governed by special enzymes called O-enzymes, which are attached to objects or identity arrows i between indistinguishable objects. An O-enzyme may also attach specific enzymes to the descendents or identity arrow between them.

There are two types, O2-enzymes attached to objects, and O1-enzymes. The objects X_i connected by identity arrows with attached O1-enzymes form clusters which are in one 1-neighborhood. Let us regard the O1-enzymes in this cluster as one O-enzyme. It makes a copy of some representative object X_i in the cluster and links it to all the X_j in the cluster by bidirectional time links. O2-enzymes at X make two descendents of X and a time link *to* one copy and another *from* the other copy. This produces a "fork in time direction". The enzyme may put a bidirectional identitity link between the two descendents at the ends of the fork prongs. Enzymes may be attached to the descendent's identity arrows in a manner determined by the O1, O2-enzymes.

We note that the action of all the O-enzymes anywhere commutes, and so their action specifies a globally well defined transformation of $\mathbf{S}_t$ into $\mathbf{S}'_t$ which consists in the growth of additional elements. At this stage, $\mathbf{S}'_t$ is only defined as a graph. We make it into a SYSTEM by extending the composition rule. This is done by specifying the equivalence relations involving new links as follows

1. time links are unitary
2. triangles made from time links and identity links are **1**.

Next we consider L-enzymes. They make links. They act at time $t + \frac{1}{2}$. Like O-enzymes they are attached to links or objects in $\mathbf{S}_t$. Their action also consists in the growth of new elements, "diagonal links". Their action at time $t + \frac{1}{2}$ makes collections of "diagonal" links l. They connect objects in $\mathbf{S}_t$ with descendents of objects in $\mathbf{S}_t$ in a manner which depends on the enzyme, on the link or object it is attached to, on the neighbourhood of this link or object in $\mathbf{S}_t$, and on the descendents of objects in this neighbourhood. It may also attach enzymes to the newly made (diagonal) links. The new links l are made by composition of links in $\mathbf{S}_t$ and time links. Furthermore, the L-enzymes may put marks on the new links which will serve as indicators of adjoint relationships to be specified later. We note that the action of all the L-enzymes anywhere commutes, since links are created depending only on what was before. Therefore the action of all L-enzymes at time $t + \frac{1}{2}$ specifies a globally well defined transformation of $\mathbf{S}'_t$ into $\mathbf{S}''_t$ which consists in the growth of new elements. $\mathbf{S}''_t$ is a SYSTEM because the new links are arrows of $\mathbf{S}'_t$.

At time $t + \frac{3}{4}$ we lift the ends (sources or targets) of diagonal links which are in $\mathbf{S}_t$ by composing them with time links to or from descendents. The time links were made such that the way to do this is unique. There results a well defined SYSTEM $\mathbf{S}_t'''$ with $Cat(\mathbf{S}_t''') = Cat(\mathbf{S}_t'')$.

Finally, at time $t+1$, one considers pairs Y_1, Y_2 of of descendents arbitrary objects such that there is some link between them. The local action at (Y_1, Y_2) consists in an examination of the totality T of links between Y_1 and Y_2 and in declaring some of them adjoints of others in a manner which depends on T and the marks on the links. How this is done must be specified a priori by specifying the meaning of marks. We note that the local action for different pairs commutes. Therefore there results a well defined SYSTEM $\mathbf{S}_t''''$ with $Cat(\mathbf{S}_t'''') = Cat(\mathbf{S}_t''')$.

The objects which are descendents of objects in $\mathbf{S}_t$ and the links between them generate $\mathbf{S}_{t+1}$

This demonstrates validity of the following

Theorem 6 (Deterministic 1. order enzymatic dynamics) *Suppose that* $\mathbf{S}_t$ *can be obtained from a* SYSTEM *without attached enzymes by attaching O-enzymes and L-enzymes (as described above) to links and objects. Then the enzymes determine a unique map* $\mathbf{S}_t \mapsto \mathbf{S}_{t+1}$ *to another* SYSTEM. *If* $\mathbf{S}_t$ *is the time t-layer of the part* $\mathbf{S}_{\leq t}$ *of a drama, then* $\mathbf{S}_{\leq(t+1)}$ *is defined.*

The making of cognitive links requires a separate consideration in order to fix the extension of the composition law to them.

Given a collections f_{21} of identity links from objects of SYSTEM $\mathbf{X}_1$ to SYSTEM $\mathbf{X}_2$, and f_{32} from $\mathbf{X}_2$ to $\mathbf{X}_3$, this defines a possibly empty collection $f_{31} = f_{32} \circ f_{12}$. If f_{21} and f_{31} define local functors, then so does f_{31}. By definition, cognitive links are functors. We define arrows as equivalence classes of paths. Equivalences are generated by equivalences of paths without cognitive links, and equivalence of two paths $(b_1, ..., b_n)$ when all b_i are cognitive links and their composition defines the same local functor for the both paths.

4 Transformation theory

This section will show in what way SYSTEM's theory is a generalization of gauge theory. Some basic concepts and tools flow from this.

In quantum mechanics, Dirac's transformation theory played an important role [17]. It rests on the fact that unitarily equivalent representations of the algebra $\mathbf{A}$ of observables in Hilbert spaces are not physically distinct.

The spectral theorem assert the existence of representations in which given commuting observables are simultaneously diagonal, i.e. act as multiplication operators on function spaces.

Similarly, isomorphic SYSTEMs are not considered distinct. Here I present representation theorems and some properties of special representations are pointed out.

I explain the notion of a *representation*. In group theory, a representation of a group G is not just a homomorphism (structure preserving map) to another group G', but it is required that G' must come with some predefined structure, and the group operations must be compatible with it. More particularly, G' must consist of linear maps of a vector space, and group multiplication must be composition of maps.

Similarly, a portrait in oil is a structure preserving map of a person. The image is supposed to consist of oil paint on canvas.

Generalizing this, representations of a SYSTEM will be defined as local functors to instances of a class of SYSTEMs which are equipped with some predefined structure, and operations like composition $\circ$ of arrows are supposed to be compatible with it. Sometimes there is additional structure (e.g. composition rules for objects) which are required to be preserved.

A representation is called semi-faithful when no two objects or links are mapped into the same object or link, faithful when the same is true of arrows.

There are many kinds of representations. The most important ones have the following classes of systems as images

- communication networks: The arrows $f : X \mapsto Y$ are maps $\mathbf{f} : \Omega_X \mapsto \Omega_Y$ of sets (or spaces with more structure) and composition is composition of maps.
- archetypes
- unfrustrated systems

Archetypes are special SYSTEMs , often with few objects.

Theorem 7 (Representation as a communication network). *Every* SYSTEM *with finitely many objects and links admits a faithful representation as a communication network, i.e. there are sets Ω_X associated with objects, and links and arrows are maps $\mathbf{f} : \Omega_X \mapsto \Omega_Y$.*

Given a path $\mathcal{C} = (b_1, ..., b_n)$ from X to Y, let $f = b_n \circ ... \circ b_1$. Then $\mathbf{f} : \Omega_X \mapsto \Omega_Y$ is called the *parallel transport along the path C.*

There is a more elaborate version of theorem 7 wherein there are separate *input spaces* A_X and *output spaces* Ω_X, objects X define maps $\mathbf{1}_X : A_X \mapsto \Omega_Y$ and links and arrows are maps $\Omega_X \mapsto A_Y$. The theorem was proven in [50]; it could be generalized to SYSTEMs with more elements. The proof is constructive, but the construction of Ω_X involves elements of the whole SYSTEM.

Theorem 8 (Principal fibre bundle representation) *Let* $\mathbf{S}$ *be a* SYSTEM *with countably many objects and links, and suppose that its links are all unitary. Then it admits a representation as in theorem 7, where* Ω_X *are copies of the gauge group* G*, and the maps commute with the right action of* G *on the* Ω_X *by group multiplication.*

Maps which commute with the right action of G amount to left multiplication with group elements.

The theorem recovers the structure of lattice gauge fields in pure lattice gauge theory under the single extra structural assumption

$$forth \circ back = identity.$$

Corollary 1 (associated vector bundle representation) *Under conditions as in theorem 8, if the gauge group* G *admits a faithful representation in a vector space* Ω*, there is a representation as in theorem 7 where* Ω_X *are copies of* Ω*, and the maps* $\mathbf{f}$ *are linear. If the linear representation is a unitary representation in a Hilbert space, then the arrows are unitary maps of Hilbert spaces.*

This is the standard construction of associated vector bundles from principal fibre bundles.

PROOF OF THEOREM 8: The graph of a SYSTEM with countably many objects and links admits a spanning tree which generates an unfrustrated SYSTEM $\mathbf{T}$. Let X_0 be its root and identitfy $G = G_{X_0}$. To every object X, there is a unique unitary arrow $\mathbf{T} \ni h_X : X_0 \mapsto X$. Associate copies of the gauge group G with objects X; they are all identified with $\Omega_{X_0} = G$ via the maps h_X. Convert arrows $f : X \mapsto Y$ into elements $\mathbf{f}_G$ of the gauge group G_{X_0} according to $\mathbf{f}_G = h_Y^* \circ f \circ h_X$. $\mathbf{f}_G$ acts on $\Omega_X = G$ by $\mathbf{f}_G(g) = \mathbf{f}_G g \in G = \Omega_Y$. In this way we construct a SYSTEM which is isomorphic to the original one and has the desired properties.

4.1 Logic

SYSTEMs with unitary links are generalizations of gauge theories. There is a quite different class of SYSTEMs (and of categories [42, 41]) where links and arrows are maps of sets which need be neither surjective nor injective. Logic belongs here.

To give an example what can be done with representations, I report some theorems on logic. They were proven by Schrattenholzer in his thesis [65]. I will not reproduce the proofs.

Definition 7 (Logical archetype) *The logical archetype is the* SYSTEM *with two objects denoted* T *(*TRUE*) and* F *(*FALSE*), and links* $e : T \mapsto F, \quad e^* : F \mapsto T, \qquad o : F \mapsto F,$
The composition law is defined by the following relations:
$e \circ e^* = \mathbf{1}_F,\ e^* \circ e = \mathbf{1}_T,\ o \circ o = o = o^*,\ o \circ e = e, \quad e^* \circ o = e^*$. *In addition there is a rule for composing objects with the help of the Scheffer stroke* $|$ *:*
$T|T = F$, $T|F = F$, $F|T = F$, $F|F = T$.

In the logical archetype and in all our logical systems, the links are interpreted as "excludes", and the special case of unitary links as "not". A pair of adjoint unitary links is graphically represented as $\sim$. The objects represent potential propositions, and the Scheffer stroke $|$ is interpreted as "neither nor" (NOR). Hence $A|A$ is interpreted as *not* A. .

Note that logically, (A excludes B) implies (B excludes A), since both are equivalent to the statement that A and B are not both true. This rule of logic says that adjoints of links should be links.

In the following, we are relaxing assumption 1 by admitting links from a composite object to its constituents and vice versa.

Define a *logical* SYSTEM as a SYSTEM without equivalence relations between paths other than possible unitarity of links, in which some composite objects $A|B$ may appear, subject to the following conditions:

1. With $A|B$ also A and B are in the SYSTEM, and there are pairs of adjoint links $A \leftrightarrow A|B \leftrightarrow B$.

2. For every object A, including composite objects $A = B|C$ there is an object $A|A$. Furthermore A and $A|A$ are linked by adjoint unitary links $\sim$ in both directions.

A logical representation of a (logical) SYSTEM is a local functor into the logical archetype, subject to the additional requirement

$$F(A|B) = F(A)|F(B). \tag{11}$$

Note that it maps every object of the SYSTEM into T or F. In this way truth values are assigned.

Theorem 9 (Schrattenholzer 1999) *In a logical representation of a logical* SYSTEM*, truth values T, F are assigned in accordance with the axioms of proposition logic.*

Conversely, let $\mathbf{S}$ *be any* SYSTEM *(not necessarily a logical one), possibly with composite objects $A|B$. If there exists an assignment of truth values which is consistent with proposition logic (with the above interpretation of objects, links and Scheffer stroke $|$), then the* SYSTEM *admits a logical representation.*

The propositional content in a logical SYSTEM is in its links. They are subject to being transformed by the moves shown in 4.1. It was proven by Schrattenholzer that these rules are complete - every deduction of proposition logic is possible with their help. Furthermore, he showed that there is also a local decision calculus. The diagrams in figure 4.1 may be regarded as compound arrows if two parallel links or arrows $f_1 : X \mapsto Y$ and $f_2 : X \mapsto Y$ can be regarded as a single arrow, cp. later (definition 8).

4.2 Frustration

In the context of the representation theorem 7, paths $C = b_1, ..., b_n$ from X to Y define parallel transport $\mathbf{f_C} : \Omega_X \mapsto \Omega_Y$, of elements in Ω_X and frustration exists if the parallel transport along different path's does not agree. Local frustration occurs when this happens for paths which stay in a neighborhood. This situation appears many times in physics under different names. (If all links are unitary, frustration exists if and only if the gauge group is nontrivial)

- *Curvature* in general relativity and in the Riemannian geometry of surfaces in 3-dimensional Euklidean space. Space time is curved in general relativity if the parallel transport of a tangent vector [e.g. a 4-velocity] from space time point X to X along two different paths need not give the same result. Generically, the gauge group is the Lorentz group.

- *Field strength* in electromagnetism and in the gauge theories of elementary particle physics, where vectors in colour spaces are parallel transported.

- *Arbitrage* in financial markets, and

- *Frustration* in spin glasses.

In a spin glass model, one has spins attached to the sites X of a lattice. They may either point up or down. There are links between some of them.

1) adjoint of a link

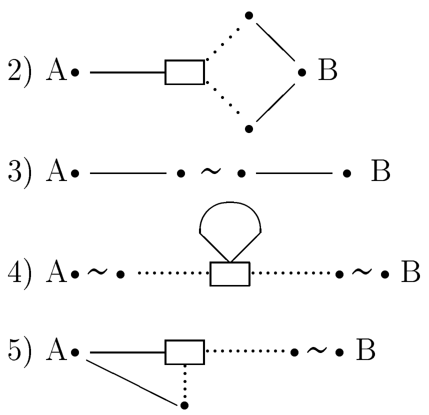

Figure 7: Schrattenholzer moves to make logical deductions. Every one of the above compound arrows between A and B may be replaced by a link pair $A - B$. Lines represent adjoint pairs of links, and $\bullet \ldots \square \ldots \bullet$ stands for a composite object $C|D$ with links to its constituents C, D; $\widetilde{\ }$ is a unitary link pair, interpreted as "is not".

They are assigned values $+1$ if it is energetically favorable for the spins at their ends to be parallel, and -1 if antiparallel is favored. There is frustration if the requirements for energetically favorable alignment of spins are in conflict. [10]

A gauge theory of financial markets was presented by Ilinski [34]

Eschers impossible pictures provide other examples of frustration: People move around and up and up, yet arrive back at their starting point. In a representation of a threedimensional scene, this is impossible, because change of height should be path independent. The pictures also illustrate the point that frustration prevents the assignment of global meaning (height) by synchronization. There is a lot of frustration in human communication. We do not exclusively communicate facts which have a globally defined meaning.

If we think of the absence of a fundamental adjoint as a kind of frustration also, [11] then in essentially all our examples nontrivial changes in time are

[10]This explains the name. In real life there is frustration if one's different desires cannot all be fulfilled because they are mutually incompatible.

[11]$forth \circ back \neq identity$ in some sense when forth is a one way street

associated with frustration. Thus, *change is caused by frustration.*

5 Managing complexity

The main theme of this section is the management of complexity by construction of simplified models, also known as effective theories, which operate on coarser scales. Thermodynamics and electrodynamics of polarizable media are well known examples of effective theories in physics.

Before coming to this I will discuss emergence as a manifestation of complexity, and some alternative strategies to deal with complexity.

5.1 What is complexity, emergence?

Emergence is sometimes described in terms like these. "From several components which happen to get together something fundamentally new originates, often with totally unexpected properties. The classical example is water whose properties are not predictable from those of hydrogen and oxygen." It is characterized as surprising behaviour which cannot be anticipated form the behaviour of their *isolated* parts [10, 21, 3]. But theoretical chemists can predict the properties of water from those of hydrogen and oxygen atoms. The mystery comes from ignoring the links, here chemical bonds.

Here I am only interested in emergence in complex systems. I consider a system as genuinely complex if it shows behavior which cannot be understood by considering small subsystems in isolation. Such behavior I call emergent. Since they do not show in small subsystems, such emergent phenomena are nonlocal phenomena. We wish to understand them as a consequence of local interactions, including those from links between constituents of different subsystems of any kind.

This is not a task which is hopeless by definition.

In quantum field theory there are several known mechanisms which lead to emergent phenomena, and string theorists are exploring more [14].

In general, there are several strategies
- large scale computer simulation
- special mechanisms
- exploitation of symmetry
- multiscale analysis

Although it is true that computers get faster much more quickly than scientists get smarter, it is not true this will solve all problems in due time. Large scale computer simulations do not qualify as a universal brute force method because the computer does not tell what to look for.

5.2 Special mechanisms

Special mechanisms of particular interest for life include those discussed in section 3.1, including the the splitFork dynamics. They are based on propagating shock waves.

Several mechanisms are known in gauge field theory which lead to nonlocal phenomena, besides propagating harmonic waves. Typically they imply protection of wave propagation against nonlinear perturbation, forbidding generation of masses.

1. Gauge invariance in gauge theories whose gauge group possesses a nontrivial center Γ. [12] This includes theories with an abelian (=commutative) gauge group like Maxwells theory. Gauss' law asserts that the presence of central charges causes flux which can be observed arbitrarily far away. Central charges come from matter fields which transform nontrivially under Γ. Electrically charged fields in Electrodynamics and quark fields in Quantum Chromodynamics are examples. In the latter example, long lines of flux cost too much energy, therefore quarks are confined and physical states carry no central charge [53].

2. Chiral invariance. The deeper reason behind this is the Atiyah Singer index theorem applied to the Dirac operator D in an external gauge field. It implies that D must have zero modes for certain boundary conditions, in numbers depending on them. This sensitivity to boundary conditions implies that the Greens function for D - the Dirac propagator - will have infinite correlation length when it exists at all, for arbitrary gauge field. Contrast the covariant Laplacian in an external gauge field with nonvanishing field strength. It is strictly positive and its Green's function decays exponentially [6]. The Atiyah Singer index theorem applies in the continuum. The approximate zero modes have also been found in computations of the spectrum of the Kogut Susskind discretization of the Dirac operator on a lattice [37].

3. Supersymmetry and additional mechanisms now under consideration in string theory cannot be discussed here.

[12] The center of a group G consists of thoses elements which commute with all elements

5.3 Exploitation of symmetry

Symmetries are properties of a system as a whole which are often rather easy to detect. When detected, they can be exploited.

A crucial problem in complex systems is often the determination of the long distance behaviour. In favorable circumstances this problem can be solved or reduced to manageable problems by exploitation of symmetries. Let us consider examples.

Propagation of waves is an emergent phenomenon because local equations of motion lead to nonlocal phenomena. Electromagnetic waves, sound waves, and matter waves are known examples. In homogeneous media, translation symmetry can be used to reduce the problem to one which is no longer complex in our sense. Consider for instance the wave function $\Psi(\mathbf{x})e^{-i\omega t}$ of noninteracting electrons in a potential which is invariant under lattice translations $\mathbf{x} \mapsto \mathbf{x} + \sum n_i \mathbf{e}_i$, $n_i \in \mathbf{Z}$. Herein, $\mathbf{e_i}(i = 1, 2, 3)$ are some given vectors which define a lattice. One must solve the 1-particle Schrödinger equation. $\mathbf{x}$ could be points of a continuous space or of a discretization of it.

One introduces Bloch waves which are invariant under lattice translations,

$$\Psi(x) = e^{i\mathbf{kx}} u_{\mathbf{k}}(x), \tag{12}$$

$$u_{\mathbf{k}}(\mathbf{x} + \mathbf{e}_i) = u_{\mathbf{k}}(\mathbf{x}), \tag{13}$$

and one is left with Schrödinger equations for $u_{\mathbf{k}}$ on a single lattice cell with periodic boundary conditions, i.e. on a compact space without "large distances".

A more subtle example is the treatment of statistical mechanical systems (in equilibrium) at a critical point. By definition, the correlation length is infinite at a critical point; therefore there are correlations between very distant regions of space, and so the system is complex in our sense to begin with. The problem is to find the long distance behavior. Under suitable conditions, the long distance behaviour can be described by a field theory which is invariant under all conformal transformations. It was shown i the seventies [47] that any such theory can be regarded as living on a compact space and there is a "conformal Hamiltonian" which has a purely discrete spectrum ≥ 0 with only a finite number of eigenstates below any finite value E. This is true in any dimension ≥ 2; in 2 dimensions one needs the extra assumption of half integral spin. The assertion about the spectrum assumes Wilson operator product expansions as asymptotic expansions (they are then automatically summable to convergent expansions [54]). On a compact space, there are no large distances any more.

Another subtle class of "manageable" systems are integrable models [23] Again, their treatment involves subtle transformations to systems which are in a sense "no longer complex". The author is not prepared to enter into a discussion of these methods, although they deserve mention here because they exploit the property of certain equations of motion that can be expressed as a requirement of no frustration.

Nearly all the standard methods of theoretical physics to deal with complex sytems are based on the exploitation of symmetries. Often one regards the system of interest as obtained by perturbing a "free" system. The free system is solved by exploitation of symmetries, and the perturbation expansion involves calculations within the framework of the free theory.

However, these methods are limited in their applicability.

5.4 Multiscale analysis

The general idea of multiscale analysis is that, although by definition complex systems cannot be understood by examining small subsystems in isolation, a complexity reduction can be achieved by doing so. It constructs objects and links of a new SYSTEM whose objects represent subsystems of the old one, but which have much fewer degrees of freedom. The new SYSTEM is still complex, but typically the procedure can be iterated. In practise, few repetitions suffice because the number of objects decreases exponentially.

The axiomatic properties of SYSTEMs are not quite suitable for the purpose of multiscale analysis, but there is a natural way to extend them without seriously violating the philosophical principle of *minimal a priori structure.*

It is natural to admit the possibility that two (or more) parallel links or arrows $b_1 : X \mapsto Y$ and $b_2 : X \mapsto Y$ are regarded as a single arrow, denoted $b_1 \oplus b_2$. I emphasize that *no assumption of linearity is involved* at this stage. In the extreme case, $\oplus$ could be a direct sum.

Definition 8 (Semiadditive System). *A semiadditive system* $\mathbf{S}^+$ *satisfies the axioms of a* SYSTEM, *except that arrows may be composed from links and their adjoints with the help of two operations,* $\circ$ *and* $\oplus$. *The* $\oplus$*-operation makes the set of all arrows with given source* X *and target* Y *into an additive semigroup. The distributive law holds*

$$(f_1 \oplus f_2) \circ (g_1 \oplus g_2) = (f_1 \circ g_1) \oplus (f_2 \circ g_1) \oplus (f_1 \circ g_2) \oplus (f_2 \circ g_2)$$

A local functor F of a semiadditive system obeys $F(g \oplus h) = F(g) \oplus F(h)$.

An arrow o is a zero arrow if $f \oplus o = f$ for all f. It is understood that arrows are modulo zero arrows.

If $f \oplus h = g \oplus h$ implies $f = g$, whatever h, then the additive semigroup admits a unique extension to an additive group.

But there are some important examples, socalled *discrete event dynamical systems* (DED's) [8] where this property does not hold. They have real (or matrix) valued links (similarly as in Maxwell theory), with addition $+$ as composition $\circ$, and $f \oplus g = max(f, g)$. In the case with one object, this is the *time table* or *Max Plus*-"algebra". For matrices, the maximum is taken entry by entry.

In the rest of this section, I work in the category of semiadditive systems. and the word *system* shall mean semiadditive system.

5.4.1 Deterministic case. Multigrid methods

For definiteness sake, I consider iterative solution of optimization problems. This is a very general class of problems. For instance, finding a solution x of an equation $f(x) = g$ is equivalent to finding a minimum of $dist(f(x), g)$ if $f(x)$ and g are in a metric space with distance $dist$.

We seek $\mathbf{S}$ in a class $\mathcal{S}$ of systems such that a given local cost function $\mathcal{H}$ is minimized. $\mathcal{H}$ assigns a real number $\mathcal{H}(S)$ to every $\mathbf{S} \in \mathcal{S}$ which is a sum of contributions from neighbourhoods. One seeks approximate solutions. Therefore a criterium for a tolerable error should also be specified.

In the computation of an iterative solution one starts from a rough approximation $\mathbf{S} = \mathbf{S}_{init}$ and one has a collection of local structural transformations (enzymes) to act on $\mathbf{S} \in \mathcal{S}$. I assume that it contains enzymes

1. to solve the local problems
2. to compose links with $\circ$ and $\oplus$.

The precise local problems depend on the problem, but they are always optimization problems for subsystems of individual neighborhoods within $\mathbf{S}$. Put another way we wish to reduce a global optimization problem to a local one.

Under the stated assumptions, there is a universal problem solving strategy, *relaxation*. One sweeps through the system and determines what is variable in individual links and objects in such a way that the local cost function is minimized subject to the constraint that everything else remains constant. Because of locality of the cost functional, this is a local problem.

When relaxation is very slow to converge, one speaks of *critical slowing down*. It is a typical effect of genuine complexity, because relaxation is inefficient in dealing with nonlocal phenomena. Multiscale analysis comes in when there is critical slowing down. The problem that one may get stuck in local minima is something else again; it is not the issue under consideration here.

In a multiscale analysis one introduces levels 0,1,2,... . The system $\mathbf{S}$ is level 0. One constructs further systems $\mathbf{S}^1, \mathbf{S}^2, ...$ called level 1,2,... and links between them. In a genuine multigrid (as opposed to unigrid) the only links between levels connect $\mathbf{S}^j$ and $\mathbf{S}^{j+1}$.

An object $X^1 \in \mathbf{S}^1$ represents a subsystem of $\mathbf{S}$, and similarly for links. $\mathbf{S}^1$ varies with $\mathbf{S}$. It is to be constructed together with a local cost function $\mathcal{H}^1(\mathbf{S}^1)$ in such a way that

1) The conditional minimum $\tilde{\mathbf{S}}(\mathbf{S}^1)$ of $\mathcal{H}(\mathbf{S})$ under the constraint that $\mathbf{S}^1$ is fixed can be found by fast converging relaxation in level 0. $\tilde{\mathbf{S}}(\mathbf{S}^1)$ will be called the optimal interpolation of $\mathbf{S}^1$.

2) If $\mathbf{S}^1$ is a minimum of $\mathcal{H}^1$ then $\tilde{\mathbf{S}}(\mathbf{S}^1)$ is an approximate minimum of $\mathcal{H}$ which can be corrected by further relaxation sweeps on level 0.

In this way the problem is reduced to minimization on level 1. To solve this problem, one introduces level 2, and so on. The technical aspects of how to organize the whole iteration scheme (V-cycles, W-cycles ...) shall not interest us here [68].

The objects and links of $\mathbf{S}^1$ may contain data which reflect the structure of the corresponding subsystems of $\mathbf{S}$, but they do not determine it uniquely. In this sense there is complexity reduction. Typically, the number of elements (links and objects) in level 1 is only a fraction of those in level 0, and the numbers of internal degrees of freedom of individual elements is about the same. The idea is that one only retains structural information to the extent that it is relevant for the cooperation of subsystems as a whole that is responsible for nonlocal emergent phenomena.

The problem with this method is the fulfillment of the above requirements 1) and 2). There are very different kinds of optimization problems, and only a part of them can be successfully treated with existing multigrid technology. Deterministic equations of motion are intractable except in favorable cases.

Let me describe an

Example 2 (discretized linear elliptic PDE's) *We seek the solution of a* *system of linear equations for vector valued functions* $u = \{u_z\}$,

$$\sum_w L_{zw} u_w + f_z = 0 \ . \tag{14}$$

or $Lu + f = 0$ *for short.* L_{zw} *are linear maps (matrices), and* f *is a given vector function (section, really ...),* $u_z, f_z \in \Omega_z$.

We assume that L is a positive operator. The system theoretic interpretation is as follows.

I work within the context of the associated vector bundle representation theorem, corollary 1, with Hilbert spaces Ω_z with scalar product $< . >$. Addition $\oplus$ of links shall be written as $+$, symbols $\circ$ are omited, and 0 is the zero link.

$\mathbf{S}$ shall consist of a constant SYSTEM $\bar{\mathbf{S}}$ with at most one link $L_{wz} : z \mapsto w$ for every pair z, w of objects, plus one object ∞ with associated vector space Ω_∞. For simplicity, admit and take $\Omega_\infty = \mathbf{R}$, possibly not isomorphic with $\Omega_z, z \in \bar{\mathbf{S}}$, and identitfy f_z, u_z with maps $\mathbf{R} \mapsto \Omega_z$, viz. $r \mapsto u_z r$. In this way, f_z and u_z become links $\infty \mapsto z$.

For every object $z \in \bar{\mathbf{S}}$, there shall be a constant link (linear map) $f_z : \Omega_\infty \mapsto \Omega_z$ and one variable link (linear map) $u_z : \Omega_\infty \mapsto \Omega_z$. The cost functional shall be quadratic,

$$\mathcal{H}(\mathbf{S}) \equiv E(u) = \frac{1}{2} \sum_{z,w} < u_w, L_{wz} u_z > + \sum_z < f_z, u_z > \ , \tag{15}$$

The local problem is the solution of some equation $L_{zz} u_z + g_z = 0$ for individual z. Write its solution as $(-L_{zz}^{-1}) g_z$, without implying that negative and inverse can be computed separately. $(-L_{zz})^{-1}$ are loops. Relaxation at z updates

$$u_z \mapsto (-L_{zz}^{-1})[u_z + \sum_{w \neq z} L_{zw} u_w].$$

This only involves solution of the aforementioned local problem, and composition and addition of links. By assumption, there are enzymes to solve it. The iterative solution is

$$u_z = (-L_{zz})^{-1} \sum_w \sum_{p: w \mapsto z} \mathbf{p} f_w \tag{16}$$

where the sum is over all paths p with links $L_{zw}(-L_{ww}^{-1})$, and $\mathbf{p}$ is the arrow associated with the path p. This illustrates once again the principle of universality and minimal a priori structure. The axiomatic composition operations $\oplus$ and $\circ$ suffice to reduce a global to a local problem, and if the local problem is a global one on a finer scale, the procedure can be repeated.

If there are important contributions in (16) from very long paths, the iteration is slow to converge, and there is critical slowing down. This happens when L has very small eigenvalues. The multigrid method deals with this. For reasons of space, I will not give details here but treat the stochastic case instead.

5.4.2 Stochastic case. Renormalization group

The setup in the stochastic case is as in the deterministic case except that now we dont want to minimize a cost function $\mathcal{H}$, but study probability distributions for systems $\mathbf{S} \in \mathcal{S}$ of the form $p(\mathbf{S}) = Z^{-1}e^{-\beta\mathcal{H}(\mathbf{S})}p_0(\mathbf{S})$ where the a priori distribution p_0 assigns equal probabilities in some sense. We are also interested in properties of the expectation values. Thus, $\mathbf{S}$ are random systems, and their elements are random variables.

In a critical situation, we expect long range correlations. Long range is measured by path length. If $\mathbf{S}$ is a drama, the long range correlations could be in time. [13]

Starting from level 0, one introduces systems $\mathbf{S}^1$ of level 1 and a probability distribution $p^1(\mathbf{S}^1)$ for them. The links and objects of level 1 represent subsystems of $\mathbf{S}$ in the same way as before, and they vary with $\mathbf{S}$. One considers the conditional probability distribution $p(\mathbf{S}|\mathbf{S}^1)$ for $\mathbf{S}$, given $\mathbf{S}^1$. One demands that

1. The conditional probability distribution $p(\mathbf{S}|\mathbf{S}^1)$ for $\mathbf{S}$ shows no long range correlations.
2. $p(\mathbf{S}) = \sum_{\mathbf{S}^1} p(\mathbf{S}|\mathbf{S}^1)p^1(\mathbf{S}^1)$.

Typically, the objects x of $\mathbf{S}^1$ contain data Φ_x which reflect the structure of the subsystems $\mathbf{X}$ of $\mathbf{S}$ to which x corresponds. It does so to the extent that it is relevant for cooperative effects, i.e. long range correlations. And similarly for links. It does not fix he SYSTEM $\mathbf{X}$ uniquely. The requirement 1 says that emergent phenomena disappear when Φ_x are frozen.

The quantities Φ_x were introduced into statistical mechanics by Kadanoff [36] under the name of *block spin*. They are also called *macros* [59]. The main problem in the approach is to find the suitable subsystems X (blocks) and

[13] Haken's slave principle [30] was invented to deal with this case

a good choice of block spins or macros. When one succeeds, the study of nonlocal phenomena - long range correlations - has been lifted to level 1, and a complexity reduction has been achieved. Now one can iterate the procedure.

Given the blocks and a choice of block spin, how are the links in $\mathbf{S}^1$ and the cost function on level 1 constructed and how can one find out whether the requirements 1 and 2 are satisfied?

No general procedure is known which is always guaranteed to work. But the example below gives an idea how the task may be performed by enzymatic computation, at least in favorable cases.

There remains the problem of how to choose the blocks and the block spins, much as in the deterministic case. In successful applications of the real space renormalization group to ferromagnets, lattice gauge theories [5], and other problems in physics, successful blocks and block spins could be guessed a priori. The big task for the future is to construct general and systematic procedures to find them. Neural nets [33] are a very difficult example of a prospective application.

For a ferromagnet in thermal equilibrium, suitable blocks are cubes in space of some extension , and a suitable block spin is the total magnetization in the cube. It fixes only the average value of the magnetic moment vectors of the elementary magnets in the cube. But this is all that matters for the purpose of determining long range correlations i.e. the physics at coarse scales [36].

The construction of block spins is a cognitive procedure. It involves construction of *new* links which are not composed from existing links alone, although very nearly so. In the construction below, the irreducibly new link is the identification of x with a representative ("typical") object in X.

In biological or social organisms we imagine that they have limited cognitive capabilities which have been acquired by evolution. They determine what blocks and blockspins are subject to being tried out. In the spirit of Ehresmann and Vanbremeersch [18], one may imagine that they carry templates of the index category $\mathbf{J}$ which is used in the construction of the block spin, cp. the example below and subsection 5.4.3.

Example 3 (Block spin) *The* SYSTEM $\mathbf{S}$ *and the cost functional* $\mathcal{H}(\mathbf{S})$ *are the same as in the deterministic case, example 2.* $u_z \in \Omega_z$ *are now random variables. Their a priori distribution is given by the uniform measure* du_z *in*

Ω_z *and we seek to examine the probability measure*

$$d\mu(u) = Z^{-1} e^{-\beta \mathcal{H}(u)} \prod_z du_z \tag{17}$$

$$\mathcal{H}(u) = \frac{1}{2} \sum_{z,w} < u_z, L_{zw} u_w > + \sum_z < f_z, u_z > \tag{18}$$

This would become (almost) a realistic Euclidean quantum field theory model of strongly interacting elementary particles if z formed a hypercubic 4-dimensional lattice and if the lattice gauge fields $L_{zw}, z \neq w$ were dynamical, with values in $SU(3)$. $d\mu$ is a Gaussian measure. [14]

Let us discuss the example. Consider subsystems $\mathbf{X}$ of $\bar{\mathbf{S}}$ (level 0), for instance a hypercube of some side length in the above mentioned 4-dimensional hypercubic lattice, together with all links L_{zw} between its objects (lattice sites) z, w. We seek to represent $\mathbf{X}$ by one object x at the next level 1, and construct a block spin $U_x \in \Omega_x$ as some average of u_z over objects $z \in X$,

$$U_x = \sum_{z \in X} C_{xz} u_z \ . \tag{19}$$

We omit $\circ$-symbols again for the composition of maps; $\sum$ is here addition in the vector space Ω_x. $C_{xz} : \Omega_z \mapsto \Omega_x$ are linear maps. Elements $u_z \in \Omega_z$ for different z are in different spaces. To add them up they need to be parallel transported to some representative site $\hat{x} \in X$ first. This can be done with the help of a tree $\mathbf{J}$ with root x which specifies a unique arrow $z \mapsto x$, i.e. a linear map $t_{\hat{x}z} : \Omega_z \mapsto \Omega_{\hat{x}}$. $\mathbf{J}$ could be a tree made of links of X or it could be a star of arrows which are sums of parallel transporter from z to $\hat{x}$ along some classes of paths. Given $\mathbf{J}$ we construct

$$C_{xz} = C_{x\hat{x}} t_{\hat{x}z} \ . \tag{20}$$

This involves one *new* link $C_{x\hat{x}} : \hat{x} \mapsto x$ which links the representative object $\hat{x} \in X$ in X to the representative x of X on the next level. Some such new link is inevitably needed since we have no links between levels to start with. We choose $C_{\hat{x}x}$ as identificaton map of $\Omega_{\hat{x}}$ and Ω_x. This completes the block spin definition.

[14] The qualification "almost" refers to the fact that u should be Fermi fields rather than true random variables, and L should be the Dirac operator in a gauge field, which is not positive

Categorical Remark 6 **J** *is an unfrustrated subcategory of* $Cat(\mathbf{X})$ *of the type of a partial order [40], p.11. At this stage we don't insist on making it into a* SYSTEM, *so adjoints of its arrows need not be in* **J**. *But if the arrows* t_{xz} *are unitary, we could put their adjoints into* **J** *and make it into a* SYSTEM *without introducing frustration. It retains the type of a preorder.*

Now we come to the construction of the links at level 1 and the examination of the locality requirements.

The standard procedure [25] is to construct an interpolation operator A i.e. a collection of links $A_{zx} : x \mapsto z,\ z \in \bar{\mathbf{S}}$ for all $x \in \mathbf{S}^1$ in such a way that

$$(LA)_{zx} \equiv \sum_w L_{zw} A_{wx} = \sum_y C^*_{yz} L^1_{yx} \tag{21}$$

for some L^1. The links A_{zx} are supposed to be composed from C^*_{xw} and arrows in $\bar{\mathbf{S}}$. The sum over objects y of level 1 has only one term if every z is in only one block y.

A suitable A may be obtained by minimizing $\sum_{z,x} < A^*_{zx}, (LA)_{zx} >$ subject to the constraint $CA = 1$, i.e. $\sum_z C_{xz} A_{zy} = \delta_{xy}$ for all x, y. This extra condition makes A unique, and if the block spin choice is "good", A_{zx} will be local in the sense that A_{zx} is very nearly zero except for z in a reasonably small neighborhood of the subsystem $\mathbf{X}$. L^1_{xy} are the desired links at the next level. Under the stated condition it will also be local - i.e. only a few L^1_{xy} will be not very nearly zero. This follows from $L^1 = CLA$ if $CC^* = 1$.

Given A, any u can be uniquely split into contribution from a blockspin $U = \{U_x\}$ and a fluctuation field $\zeta = \{\zeta_z\}$ which satisfies the constraint $C\zeta = 0$ so that it contributes nothing to the blockspin.

$$u_z = \zeta_z + \sum_x A_{zx} U_x. \tag{22}$$

The cost function decomposes as

$$\mathcal{H}(u) = \frac{1}{2} < \zeta L \zeta > + < f, L\zeta > + \frac{1}{2} < U, L^1 U > + < Cf, L^1 U > . \tag{23}$$

The constraint $CA = 1$ may be put into the measure $d\mu$ in the form of a δ-function which is the limit of a Gaussian. One finds that ζ are Gaussian random variables [27] with covariance $\Gamma = \lim_{\kappa \mapsto \infty} \Gamma^\kappa$, $\Gamma^\kappa = (L + \kappa C^* C)^{-1}$. One may opt to keep κ finite, thereby relaxing $CA = 1$. In this case $A = \kappa \Gamma_\kappa C^*$.

Then all locality requirements are fulfilled if Γ^{κ}_{zw} decays fast with pathdistance between z and w. In particular, A_{zx} and L^1_{xy} will also be local.

In conclusion, one needs to find a local "interpolation operator" $A = \{A_{zx}\}$ such that eq.(21) holds for some L^1. This yields the links L^1_{xy} of the SYSTEM $\mathbf{S}^1$ at the next scale.

5.4.3 (Co)Limits

Here I wish to establish the connection with the work in mathematical biology of Ehresmann and Vanbremeersh [19]. They propose to consider the objects X in level $j+1$ which represent subsystems of the level j SYSTEM as *limits* in a category. The same construction of composite objects as limits is also used in information science in what is called integration [20]. I will argue that (co)limits serve the same purpose as blockspins, and prove that the blockspin of section 5.4.2 defines a colimit.

I recall the notion of a limit [40]. Given a category $\mathbf{C}$ and a (small) category $\mathbf{J}$, called the indexing category, a functor $F : \mathbf{J} \mapsto \mathbf{C}$ is called a *diagram in* $\mathbf{C}$ *of type* $\mathbf{J}$. To be intuitive, Vanbremeersh and Ehresmann call it a *pattern of linked objects*. By the map, some objects $F_j = F(j)$ of $\mathbf{C}$ are indexed by objects j of $\mathbf{J}$.

An object L of $\mathbf{C}$ together a collection of arrows $\pi_j : L \mapsto F_j$, one for each $j \in J$, is called a *cone* $\pi : L \mapsto F$ on the diagram F of type $\mathbf{J}$ with vertex L if the following compatibility condition is satisfied. For any arrow $u : i \mapsto j$ in $\mathbf{J}$,

$$\pi_k = F(u) \circ \pi_j. \tag{24}$$

In ref. [19], $\pi = \{\pi_j\}$ is called a *collective link*. It links L to the subsystem $\mathbf{L}$ which it is supposed to represent. In our applications, the subsystem has F_j as its objects, but in general it has more arrows than $F(u), u \in \mathbf{J}$. The images of links $u \in \mathbf{J}$ are special links, which are "important for the collaboration". Note that the collective link projects out any frustration in the image of $\mathbf{J}$ in the following sense. Given two arrows $F(u_1) : F_i \mapsto F_j$ and $F(u_2) : F_i \mapsto F_j$, eq.(24) implies $F(u_1) \circ \pi_j = F(u_2) \circ \pi_j$ .. If F maps several j to the same object $z = F(j)$, we need only one link $\pi_j \equiv \tilde{\pi}_z : L \mapsto z$ for them, and we may regard $\tilde{\pi}_z$ as a (link-) field whose argument is $z \in \mathbf{L}$. The condition (24) can be interpreted to say that this field is constant under parallel transport along paths which are images of paths in $\mathbf{J}$. We say that it is "constant along $\mathbf{J}$" for short.

A cone $\pi : L \mapsto F$ is called a *limit of the diagram* $\mathbf{J}$ if the following uniqueness property holds. Given any other cone $f : Y \mapsto F$, there exists a *unique* arrow $g : X \mapsto L$ such that $f_j = \pi_j \circ g$. By abuse of language, L is called the limit. The condition of a cone f means that $\tilde{f}_z$ is constant along $\mathbf{J}$.

Colimits are dual to limits, i.e. all arrows are reversed. In our applications, all categories except the index categories $\mathbf{J}$ will be SYSTEMs. Therefore $\{\pi_j\}$ defines a limit if $\{\pi^*\}$ defines a colimit of a dual diagram of type $\mathbf{J}^{op}$, where $\mathbf{J}^{op}$ is $\mathbf{J}$ with arrows reversed [40].

Theorem 10 (Blockspins as colimits) *The blockspin construction of section 5.4.2 based on a tree* $\mathbf{J}$ *defines a colimit* $x \in \mathbf{S}^1$ *in the category which contains* $Cat(\mathbf{S})$, $Cat(\mathbf{S}^1)$ *and the links* $C_{x,z} : z \mapsto x$, $(z \in \mathbf{S})$, *and in any category containing it.*

The interpretation of blockspins as (co)limits serves to translate from a quantitative description to a structural one. Before we proceed to the easy proof of theorem 10, let us discuss how the interpolation operator is interpreted. Equation (21) requires that for every x,

$$\tilde{f}_z = (LA)_{zx} = \sum_w L_{zw} A_{wx} : x \mapsto z \tag{25}$$

is constant along $\mathbf{J}$, because this is true of $\tilde{\pi}_z = C^*_{zy}$. In other words, $\{(LA)_{zx}\}$ is a collective link which defines a cone on the same diagram as for the collective link $\{C^*_{zx}\}$.

If every object z is in only one block $\hat{z}$, then $(C^* L^1)_{zx} = C^*_{z\hat{z}} L^1_{\hat{z}x}$ (no sum). Therefore, if LA defines a cone, the existence of L^1 is assured by the property that the collective link $\{C^*_{zx}\}$ defines a limit.

In conclusion, the existence of the interpolation operator requires the existence of another "factorizing" cone on the same diagram as the limit cone. The factorization into L and some A is expressed by eq.(25), $L = \{L_{zw}\}$ is the collection of links of the SYSTEM $\mathbf{S}$ at the fine scale. (Note that this is not of the form of standard factorization properties in category theory because of the sum. It involves the $\oplus$-operation of semiadditive systems.) In addition, the locality properties laid down in previous subsections should be satisfied. This means that it must be possible to approximate A_{zx} by zero if z is not in a reasonably small neighborhood of the subsystem $\mathbf{X}$ represented by x. How to go about treating the error made in this way is a subtle issue [66] which I am not prepared to discuss here.

Let us turn to the proof of the theorem. The definition of a limit is external in the sense that one needs to seek an arrow in the whole category and show its uniqueness. This is typical of the universal constructions of category theory. This feature is what makes category theory into "abstract nonsense". But there are instances where limits are internal, i.e. require only examination of the arrows which are involved in their construction. Products in Ab-categories are examples. They are necessarily biproducts, and biproducts are easy to characterize internally, cp. theorem 2 of section VIII in [40].

Products are limits with index categories $\mathbf{J}$ which have no arrows other than the identity arrows. A similar situation holds when $\mathbf{J}$ is a tree or preorder.

Lemma 1 (Colimits on trees) *Let $\mathbf{J}$ be a tree with root r, so that it is unfrustrated and there is a unique arrow $t_j : j \mapsto r$ in $\mathbf{J}$ for every object $j \in \mathbf{J}$. Then a cocone $\pi : A(\mathbf{J}) \mapsto L$ is a colimit if π_r has an inverse. Conversely, the colimit property requires that π_r has a left inverse i_r, viz $i_r \circ \pi_r = \mathbf{1}_L$.*

The dual statement is true for limits.

PROOF of theorem 10. $\mathbf{J}$ and $F(\mathbf{J})$ are identified in section 5.4.2. The collective links are $C_{xz} : z \mapsto x$, $(z \in A(\mathbf{J}))$, and the root is $\hat{x}$. The compatibility condition for a cone is satisfied by construction. The theorem is an immediate consequence of the lemma, since $C_{x\hat{x}}$, which substitutes for π_r, was chosen as an identification map, whose adjoint is its inverse. q.e.d.

PROOF of lemma 1. Given another cocone f, $g = f_r \circ i_r$ is the required unique map. It is unique because $f_r = g \circ \pi_r = h \circ \pi_r$ implies $g = h$ by invertibility. Conversely, $t = t_j$ is a cocone with vertex r and the required unique map g must be a left inverse of π_r.

Definition 9 [Ehresmann Vanbremeersch] *A* hierarchical system *is a category* $\mathbf{H}$ *whose objects are divided into* levels, *numbered* $0, 1, ..., p$, *such that each object of level $n+1$ (where $n < p$) be the limit in* $\mathbf{H}$ *of a pattern A of linked objects [=diagram] of level n (i.e. each A_i has level n).*

I propose to substitute "SYSTEM" $\mathbf{H}$ for "category $\mathbf{H}$", and count the arrows π_i in the collective links as links. I would also prefer to speak of colimits in place of limits.

If indeed blockspins and colimits are basically the same, as is suggested by the above theorem 10, this setup corresponds with the multiscale analysis of section 5.4.

It is always possible to extend categories by adding objects which represent limits of certain diagrams [18]. Mathematicians often speak of categories which have all finite limits (i.e. limits for all finite diagrams). But to do so would be contrary to the intended complexity reduction.

5.5 Dynamics on coarser levels

The somewhat abstract considerations of section 5.4.3 serve to convert block-spin constructions from the quantitative description that is used in quantum field theory and statistical mechanics to a structural description.

The problem is now how to extend the dynamics from level 0 to the higher levels. We think of a stochastic dynamics. In autopoietic systems, the objects in the higher levels will typically represent functional units of objects of lower levels which should be capable of making their elements. They may disappear, i.e. die. They may live on, possibly adapting or differentiating. And new ones may form, for instance as newly made copies of already existing units, building blocks being absorbed from the environment. How and when does this happen?

Diagrams of type **J** were identified with certain types of block spins, and the object x to which the block spin is attached was identified with a limit of the diagram. The objects x represent subsystems **X** of **S** whose constituents cooperate through links (channels of communication) which are determined by the diagram. x may stand for organs in a biological organism, for institutions of a society, for extended domains in a ferromagnet, or for any kind of a "thing". We only want to keep or acquire them in our model when they achieve something or are needed to achieve something, namely the cure of locality problems as discussed in sections 5.4.1,5.4.2. This is the criterium by which objects representing subsystems will appear or disappear.

An object x of this kind in level $n+1$ may disappear if the limit of a diagram in level n ceases to exist. This mechanism was suggesed by Ehresmann and Vanbremeersh. It can happen when the diagram is disrupted because the links involved in the collaboration disappear. I give an example in a moment. It may also cease to exist because frustration appears in the diagram, so that the cone ceases to exist. Informally speaking, confusion arises because communication of the collaborators through different channels produces different messages. (If **J** is a tree, this cannot happen).

It may occur that there are short range correlations only on level n in the vicinity of some subsystem **X**, even without any blockspin constraint.

No block x on level $n+1$ is needed in that case. If it is present anyway, it will have no important links L_{xy}^{n+1} to other objects y on level $n+1$, hence no relations to anything in the "rest of the world $\mathbf{S}^{n+1}$". Such an object does not exist for the "rest of the world" and should be discarded.

Conversely, if locality properties in $\mathbf{S}$ or in time start to be violated, they need to be salvaged by a introducing a blockspin constraint as discussed in sections 5.4.1,5.4.2, and with the block spin comes an object to which it is attached. Suppose that the correlations are not short ranged at level n without block spin constraint. This can be decided on the basis of relaxation sweeps, i.e. by enzymatic computation. They will produce correlations growing beyond the allowed range. In this case, suitable blocks and block spin constraints will have to be introduced until the residual correlations under the block spin constraint are short ranged. If the multiscale analysis were done on the drama, short ranged would mean in particular short ranged in time. Thus, the effect of particular properties (initial conditions) which are independent of the value of the blockspins will die out quickly. Only the cooperative effects which are well described by the block spins will survive.

Here is the example for the loss of a diagram. It is a frequent cause of death that an organism or a functional part of it gets digested by a predator or parasite. Digestion is performed by special enzymes. For instance, T4 bacteriophage's nuclease enzymes degrade its *E.coli* host's chromosome (but not its own genome). Consider one organism $\mathbf{P}$ being digested by another one who attacks it by acting on its object X with its digestion enzyme according to the model mechanism of section 3.1.2. Take any subsystem $\mathbf{Q}$ of $\mathbf{P}$ which does not contain X. It will be totally disconnected after the digestion process. Therefore, the supposed images of links in the diagram $\mathbf{J}$ - the links which are essential for the collaboration - are missing. So the loss of structure at some level, e.g. by digestion, may lead to loss of the limit object in the next level.

The choice of block spin will typically not be unique. But if suitable block spins can be found, the long range correlations are under control, and therefore all emergent phenomena. They are merely described in a different language when different block spins are chosen.

What has been proposed here is a reductionist scenario - the dynamics on the lower level determines what happens at higher levels, modulo switch of language. Locality is the crucial ingredient of the construction. Earlier multi-level analyses of biological systems [19, 3, 21] had no locality principle and they relaxed on reductionism. Extreme views were expressed by Laughlin and

Pines. They claimed that neither life nor high temperature superconductivity can be understood from basic principles [39].

Autopoietic systems make their own elements. This appears to require a top down action of objects x at level $n+1$ to make elements which are objects or links at level n inside the subsystem $\mathbf{X}$ to which x corresponds. We understand this now. There is a dynamics at level n which gives rise to nonlocal - therefore emergent - phenomena. And these are effectively described with the help of objects x at level $n+1$.

I add few words on funtionality. Technically most convenient would be a multiscale analysis of the drama. In this way, the higher levels would also acquire longer time scales, and function would appear as structure. For intuition's sake, I spoke here of changes in time instead. Therefore we needed to speak of functionality separately. In the present approach, functionality depends on the presence of a suitable complement of enzymes. The integrity of boundaries of subsystem may also be important in order to confine the domain where enzymes act, and also the presence of channels of communication which transfer quantities of material constituents. All this should be reflected at the coarser scale when the functionality is important for cooperation at that scale.

Production of copies is an emergent phenomenon, as we saw in section 3.1.1. Typically the copy process absorbs objects which involve, at a still lower level, materials (including energy) which are conserved or supplied by the environment in limited quantities. In this way, a competition for scarce resources results which drives evolution.

Imagine a subsystem $\mathbf{X}$ is copied by the splitFork dynamics because a sufficient collection of microenzymes is present in $\mathbf{X}$. Then the splitFork-enzyme, now considered as one entity, should be attached to the object x which corresponds to $\mathbf{X}$ at the next scale.

Let me emphasize that a general block spin procedure can be much more complicated than for a ferromagnet, where the Kadanoff construction furnishes one block spin definition for all purposes and all scales. In general, the appropriate *kind* of blockspin on level $n+1$ may depend on the *values* of variables or block spins φ on level n. Since φ are random variables, there may be nonvanishing probabilities for the appropriateness of several kinds of block spin descriptions. This can lead to bifurcations. The situation on the next scale may be the same again, and a proliferation of possibilities may result. One cannot expect to find giraffes by enumeration of all possibilities of living organisms. Moreover, scaling laws (power laws) are expected to emerge in special circumstances only. Ferromagnets, the Bak Sneppen model of evolution [4] and Lotka-Volterra-models [67] are examples, but power laws are not a general indicator of criticality.

5.6 Deductive vs. observational emergence

Baas [3] makes a distinction between deducible and observational emergence. He interpretes Gödels incompleteness theorem in logic as a case of observational emergence.

In the present frameork, we may classify as *deduction* in this sense anything that involves composition of existing links (with $\circ$ and $\oplus$). This does not change the category. Deductions in proposition logic are of this kind, cp. section 4.1, figure 4.1. Observational emergence would then involve the making of cognitive links. They are new links added to the category. The blockspin constructions involve new links.

6 Semicommutative differential calculus and geometry on SYSTEMS

In this section I want to bring SYSTEM's theory closer to the traditional approach in physics which is based on differential calculus.

Given a SYSTEM **S**, there is a unique local functor to an unfrustrated SYSTEM **B** which shares the objects with **S** and inherits equivalence classes of its parallel links. In **B**, different links of **S** with the same source and target are identified, and equally the arrows. This is an example of the possible identifications mentioned in section 2.2. **B** is determined by the graph of **S** and shall be called the *base* SYSTEM of **S**, or *base* for short. I assume for simplicity that it has at most countably many elements [15].

Differential calculus and geometry puts more structure on a given base, thereby creating SYSTEMs **S** with *given* base, and it constructs algebras and modules to go with them. Changes in connectivity of the base are outside the scope of calculus.

[15] **B** need not be like a grid. For instance, it might contain grid-like subsystems together with arbitrary refinements of these. The floating lattices in the Ashtekar approach to gravity are like this [2]

The appropriate version of differential calculus to fit onto arbitrary unfrustrated SYSTEMs $\mathbf{B}$ is a special case of noncommutative differential calculus and geometry [12] which was developped by Dimakis and Müller-Hoissen [15]. I call it semicommutative because the "algebra of functions" $\mathcal{A}$ remains commutative. In this framework, conventional notions of locality and the notion of a point, which are given up in fully noncommutative differential calculus, retain their meaning. One usage is in lattice gauge theory on a hypercubic lattice. All the familiar formulae from gauge theory in the continuum remain literally true, except for the commutation relations of differentials with functions [16], cp. eq.(31).

The use of this device is in the spirit of the strategy to bring proven methods of theoretical physics to bear on very general complex systems. The discrete calculus substitutes for and is in many ways like calculus on manifolds.

Given a base $\mathbf{B}$, let $\mathcal{A}$ be the algebra with unit element consisting of real or complex functions on $\mathbf{B}$ with pointwise multiplicaton. It has a basis $\{e_X\}$ labelled by objects X of $\mathbf{B}$ such that

$$e_X e_Y = \delta_{XY} e_X \tag{26}$$

δ_{XY} being the Kronecker δ -function

From now on, the symbol $X \mapsto Y$ shall mean that there exists a link from X to Y. Since there is at most one such link b we may write $b = (XY)$. The differential algebra Ω is generated by the algebra $\mathcal{A}$ of functions and differentials e^{XY} attached to the links of $\mathbf{B}$. $\Omega_1 = span_{\mathbf{C}}\{e^{XY} : X \mapsto Y\}$ is made into a $\mathcal{A}$-bialgebra via

$$e^Z e^{XY} = \delta^{ZX} e^{XY}, \ e^{XY} e^Z = \delta^{YZ} e^{XY}. \tag{27}$$

The quantity $\rho = \sum e^{XY}$ (sum over all links) is introduced, and exterior differentiation d is defined by

$$de^X = \rho e^X - e^X \rho, \tag{28}$$

$$de^{XY} = \rho e^X \rho e^Y - e^X \rho^2 e^Y + e^X \rho e^Y \rho \tag{29}$$

and the standard graded Leibniz rule.

There are relations between the generators in the algebra. Whenever the link from X to Y is missing, $e^X \rho e^Y = 0$. Applying d implies the constraint $e^X \rho^2 e^Y = 0$, i.e.

$$\sum_Z e^{XZ} e^{ZY} = 0 \ . \tag{30}$$

It was shown by Dimakis and Müller Hoissen that this calculus reduces to something looking familiar on an "oriented" d-dimensional hypercubic lattice of lattice spacing a. Of the two directions $\pm\mu$, one is distinguished as positive, say $+\mu$, $\mu = 1, ..., d$, and links are put between nearest neighbours denoted x and $x+\mu$ in positive direction only. If x^μ are the standard coordinate functions, one computes $dx^\mu = e^{x,x+\mu}$ and therefore $df(x) = a^{-1}[f(x+\mu) - f(x)]dx^\mu$.

The constraints among differentials reproduce the standard relations $dx^\mu dx^\nu + dx^\nu dx^\mu = 0$ There is a Hodge * -operator, and the calculus shares all the properties of the continuum calculus, except that

$$f(x)dx^\mu = dx^\mu f(x+\mu) \tag{31}$$

This rectifies the Leibniz rule. To understand the usefulness of all this, consider

Theorem 11 (Gauss constraint) *The Maxwell dynamics of a free electromagnetic field in discrete space and time preserves the Gauss constraint.*

Proof: One transscribes the standard proof. Let d_s be the exterior derivative in space, $*$ the Hodge star operator in space and $d_s^* = *d*$. The electric field defines a 1-form $E = \sum E_i dx^i$. The Gauss law says that $d_s^* E = 0$. The magnetic field defines a 2-form B and the equations of motion says that $\dot{E} = d_s^* B$. Since $d^{*2} = 0$ it follows that $d_s^* \dot{E} = 0$, and so the Gauss constraint is preserved. All the quantities have their analog in the semicommutative calculus on an oriented cubic lattice, $\dot{E}$ becomes the finite difference derivative, and the equations of motion and Gauss constraint retain their form. So the proof carries over.

Acknowledgement

I would like to thank A. Brandt, I. Cohen, M. Meier-Schellersheim, S. Solomon, J. Wuerthner and Y. Xylander for many helpful and stimulating discussions.

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